On density of horospheres in dynamical laminations *

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Contents

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1 Introduction and main results

1.1 Introduction, brief description of main results and the plan of the paper

In 1985 D.Sullivan [17] had introduced a dictionary between two domains of complex dynamics: iterations of rational functions $f(z) = \frac{P(z)}{Q(z)} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ on the Riemann sphere and Kleinian groups. The latter are discrete subgroups of the group of conformal automorphisms of the Riemann sphere. This dictionary motivated many remarkable results in both domains, starting from the famous Sullivan's no wandering domain theorem [17] in the theory of iterations of rational functions.

One of the principal objects used in the study of Kleinian groups is the hyperbolic 3- manifold associated to a Kleinian group, which is the quotient of its lifted action to the hyperbolic 3- space $\mathbb{H}^3$. M.Lyubich and Y.Minsky have suggested to extend Sullivan's dictionary by providing an analogous construction for iterations of rational functions. For each rational function f they have constructed a *hyperbolic lamination* $\mathcal{H}_f$ (see [13] and Subsection 1.3 below). This is a topological space foliated by hyperbolic 3- manifolds (some of them may have singularities) so that

- a neighborhood of a nonsingular point is fiberwise homeomorphic to the product of a part of the Cantor set and 3- ball;

- the hyperbolic metric of leaves depends continuously on the transverse parameter;

- there exists a natural projection $\mathcal{H}_f \to \overline{\mathbb{C}}$ under which the (non-bijective) action $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ lifts up to a homeomorphic action $\hat{f} : \mathcal{H}_f \to \mathcal{H}_f$ that maps leaves to leaves isometrically;

- the lifted action $\hat{f}$ is proper discontinuous, and hence, the quotient $\mathcal{H}_f/\hat{f}$ is a Hausdorff topological space laminated by hyperbolic 3- manifolds (called *the quotient hyperbolic lamination*);

- the lamination $\mathcal{H}_f$ either is minimal itself (i.e., each leaf is dense), or becomes minimal after removing a finite number of isolated leaves (which may exist only in very exceptional cases); the complement to the isolated leaves will be denoted $\mathcal{H}'_f$.

The hyperbolic lamination $\mathcal{H}_f$ is constructed as follows. Take the natural extension of the dynamics of f to the space of all its backward orbits. The latter space always contains some Riemann surfaces conformally-equivalent to $\mathbb{C}$. The union of all these surfaces (denoted by $\mathcal{A}^n_f$) is invariant under the lifted dynamics. Pasting a copy of the hyperbolic 3- space to each surface, appropriate strengthening of the topology and completion of the new space thus obtained yields the hyperbolic lamination $\mathcal{H}_f$.

Recent studies of the hyperbolic 3-manifolds associated to Kleinian groups resulted in resolving of all big problems of the theory, including a positive solution of the famous Ahlfors measure conjecture (with contributions of many mathematicians, see the papers [1], [3] and references therein). On the other hand, very recently the analogous conjecture in the theory of rational iterations was proved to be wrong: by using a completely different idea proposed by A.Douady, X.Buff and A.Cheritat have constructed examples of quadratic polynomials with Julia sets of positive measure [2].

There is a hope that similarly, hyperbolic laminations associated to rational functions would shed new light on the underlying dynamics.

One of the main problems in the theory of rational iterations is the Fatou conjecture: *is it true that each critical orbit of a generic rational function of a given degree either is periodic itself, or converges to an attracting or super-attracting periodic orbit?* It is open already for the quadratic polynomials, for which it is equivalent to the No Invariant Line Field Conjecture: *is it true that there are no measurable invariant line fields supported on the Julia set?*

The answer to the latter conjecture is known to be positive for the critically-nonrecurrent rational functions without parabolics that are different from the Lattès examples. Elementary geometric proofs using laminations may be found in [13] (proposition 8.9) and [7] (subsection 4.3).

There is a hope that the laminations could be helpful in studying the No Invariant Line Field Conjecture.

The present paper studies the arrangement of the horospheres in the quotient hyperbolic lamination $\mathcal{H}_f/\hat{f}$. Let us recall their definition. The hyperbolic space with a marked point "infinity" on its boundary Riemann sphere admits a standard model of half-space in the Euclidean 3- space. Its isometries that fix the infinity are exactly the extensions of the complex affine transformations of the boundary (we call these extensions "affine isometries"). A horizontal plane (i.e., a plane parallel to the boundary) in the half-space is called a *horosphere*. The affine isometries transform the horospheres to the horospheres. The *horospheres of a quotient of* $\mathbb{H}^3$ by a discrete group of affine isometries are the quotient projection images of the horospheres in $\mathbb{H}^3$. All the horospheres mentioned above carry natural complex affine structures (they may have conical singularities) and foliate the ambient hyperbolic manifold (orbifold).

The leaves of the laminations $\mathcal{H}_f$ and $\mathcal{H}_f/\hat{f}$ are also quotients of $\mathbb{H}^3$ by discrete groups of affine isometries. Thus, all their leaves are foliated by well-defined horospheres.

The arrangement of the horospheres in the above hyperbolic laminations is related to the behavior of the modules of the derivatives of the iterations of the rational function.

The vertical geodesic flow acts on $\mathbb{H}^3$ by translating points along the geodesics issued from the infinity. This yields the leafwise vertical geodesic flows acting on $\mathcal{H}_f$ and $\mathcal{H}_f/\hat{f}$, for which the horospheric laminations are the unstable laminations.

The classical results concerning the geodesic flows on compact hyperbolic surfaces say that the horocyclic lamination is minimal [9] and uniquely ergodic [5, 16]. Their generalizations have found important applications in different domains of mathematics, including number theory. We hope that studying the vertical geodesic flow and the horospheric lamination would have applications in understanding the underlying dynamics.

The main results are stated in 1.5 and proved in Sections 2-5. The principal one (Theorem 1.49 proved in Section 2) says that at least some horospheres are always dense in $\mathcal{H}'_f/\hat{f}$, provided that the mapping f does not belong to the following list of exceptions:

$$z^{\pm d}, \text{ Chebyshev polynomials, Lattès examples.} \tag{1.1}$$

Namely, all the horospheres "over" the so-called branch-nonexceptional repelling periodic orbits are always dense.

Remark 1.1 For any f from the list (1.1) each horosphere in $\mathcal{H}_f/\hat{f}$ is nowhere dense. This follows from a result of [11] (see Corollary 1.48 in 1.5).

Theorem 1.51 (proved in 1.5) says that all the horospheres are dense in $\mathcal{H}'_f/\hat{f}$, if f does not belong to the list (1.1) and is critically-nonrecurrent without parabolic periodic points (e.g., hyperbolic). In the case, when f does not belong to (1.1), is critically-nonrecurrent and has parabolic periodic points, Theorem 1.52 (also proved in 1.5) says that all the horospheres are dense in $\mathcal{H}'_f/\hat{f}$, except for the horospheres "related" to the parabolic points. To prove density of a horosphere in Theorem 1.52, we show (Theorem 1.53 stated in 1.5 and proved in Section 3) that it accumulates to some horosphere over appropriate branch-nonexceptional repelling periodic orbit. (The limit horosphere is dense by Theorem 1.49.) Theorem 1.54 (stated in 1.5 and proved in Section 4) deals with an arbitrary rational function having a parabolic periodic point. It says that each horosphere in a leaf associated to this point is closed in $\mathcal{H}_f/\hat{f}$ and does not accumulate to itself.

Remark 1.2 While the present paper was in preparation, Theorem 1.51 has already been applied to prove the unique ergodicity of the quotient horospheric laminations associated to appropriate rational functions. The latter include all the hyperbolic and critically-finite functions [8]. The results of the paper [8] with brief proofs are announced in [7].

Remark 1.3 There exist (even hyperbolic) rational functions that do not belong to the list (1.1) such that the corresponding hyperbolic lamination $\mathcal{H}_f$ has a leaf whose horospheres are nowhere dense in $\mathcal{H}_f$. This is true, e.g., for the quadratic polynomials $f_\varepsilon(z) = z^2 + \varepsilon$ with $\varepsilon \in E = (-\infty, \frac{1}{4}) \setminus \{0, \ -2\}$, and also with ε belonging to a complex neighborhood of the set E (Theorem 1.57 and its Addendum, both stated in 1.5 and proved in Section 5). The above leaf with nondense horospheres is associated to a repelling fixed point (that is real, if so is ε).

Example 1.4 Consider the above quadratic polynomial family $f_\varepsilon(z) = z^2 + \varepsilon$. It is well-known that the quotient laminations $\mathcal{H}_{f_0}/\hat{f}_0$ and $\mathcal{H}_{f_\varepsilon}/\hat{f}_\varepsilon$ are homeomorphic for all $\varepsilon \neq 0$ small enough. The homeomorphism sends leaves to leaves but not isometrically. On the other hand, Theorem 1.51 implies that if $\varepsilon \neq 0$ is small enough, then each horosphere in the latter lamination is dense, while no horosphere in the former one is dense (Corollary 1.48).

The necessary background material is recalled in Subsections 1.2 (rational iterations), 1.3 (affine and hyperbolic laminations) and 1.4 (horospheres and their metric properties).

For the proof of Theorem 1.49 we fix a horosphere in $\mathcal{H}_f$ "over" a branch-nonexceptional repelling periodic orbit and show that the union of the images of the horosphere under the forward and the backward iterations of $\hat{f}$ is dense in $\mathcal{H}'_f/\hat{f}$. To do this, we study the holonomies of the horosphere along loops based at a repelling periodic point. We show that the images of a point of the horosphere under subsequently applied dynamics and holonomies are dense in the projection preimage of the base point. To this end, we use the description of the holonomy in terms of basic cocycle (its definition and some basic properties are recalled in Subsection 1.4).

Some results of the paper with brief proofs were announced in [6].

Earlier some partial result on density of horospheres was obtained in a joint work by M.Yu.Lyubich and D.Saric [14] (under additional assumptions on the arithmetic nature of the multipliers at the repelling periodic points).

Everywhere below we assume that the rational function $f = \frac{P(z)}{Q(z)} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ under consideration has degree at least 2.

1.2 Background material 1: rational iterations

The basic notions and facts of holomorphic dynamics recalled here are contained, e.g., in [12] and [13]. Let

$$f = \frac{P(z)}{Q(z)} : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \text{ be a rational function. Recall that}$$

- its *Julia set* $J = J(f)$ is the closure of the union of the repelling periodic points, see the next definition. An equivalent definition of the Julia set says that its complement $\overline{\mathbb{C}} \setminus J$ (called the *Fatou set*) is the maximal open subset where the iterations f^n form a normal family (i.e., are equicontinuous on compact subsets). One has

$$f^{-1}(J) = J = f(J). \tag{1.2}$$

Definition 1.5 A germ of nonconstant holomorphic mapping $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ at a fixed point 0 is called *attracting (repelling / parabolic, superattracting)*, if its derivative at the fixed point respectively has nonzero module less than 1 (has module greater than 1 / is equal to a root of unity and no iteration of the mapping f is identity / is equal to zero). An attracting (repelling, parabolic or superattracting) periodic point of a rational mapping is a fixed point (of the corresponding type) of its iteration.

Definition 1.6 A rational function is said to be *hyperbolic*, if the forward orbit of each its critical point either is periodic itself (and hence, superattracting), or tends to an attracting (or a superattracting) periodic orbit.

Definition 1.7 Given a rational function. A point of the Riemann sphere is called *post-critical*, if it belongs to the forward orbit of a critical point. A rational function is called *critically-finite*, if the number of its postcritical points is finite.

Definition 1.8 The ω- limit set $\omega(c)$ of a point $c \in \overline{\mathbb{C}}$ is the set of limits of converging subsequences of its forward orbit $\{f^n(c) | n \geq 0\}$ (the ω- limit set of a periodic orbit is the orbit itself). A point c is called *recurrent*, if $c \in \omega(c)$.

Definition 1.9 A rational mapping is called *critically-nonrecurrent*, if each its critical point is either nonrecurrent, or periodic (or equivalently, each critical point in the Julia set is nonrecurrent).

Example 1.10 The following mappings are critically-nonrecurrent: any hyperbolic mapping; any critically-finite mapping; any quadratic polynomial with a parabolic periodic orbit. A hyperbolic mapping has no parabolic periodic points.

Theorem 1.11 *([12], p.60) A germ of conformal mapping at an attracting (repelling) fixed point is always conformally linearizable: there exists a local conformal coordinate in which the germ is equal to its linear part (the multiplication by its derivative at the fixed point).*

Remark 1.12 Let $f(z) = z + z^{k+1} + \dots$ be a parabolic germ tangent to the identity. The set $\{z^k \in \mathbb{R}_+\}$ consists of k rays going out of 0 (called *repelling rays*) such that

- each repelling ray is contained in appropriate sector S (called *repelling sector*) for which there exists an arbitrarily small neighborhood $U = U(0) \subset \mathbb{C}$ where f is univalent and such

that $f(S\cap U)\supset S\cap U$ and each backward orbit of the restriction $f|_{S\cap U}$ enters the fixed point 0 asymptotically along the corresponding repelling ray;

- there is a canonical 1-to-1 conformal coordinate t on $S\cap U$ in which f acts by translation: $t\mapsto t+1$; if the above sector S is chosen large enough, then this coordinate parametrizes $S\cap U$ by a domain in $\mathbb{C}$ containing a left half-plane; the above coordinate is well-defined up to translation and is called *Fatou coordinate* (see [4], [19]).

For any parabolic germ (not necessarily tangent to the identity) its appropriate iteration is tangent to the identity. By definition, the repelling rays and sectors of the former are those (defined above) of the latter.

Let us recall what are Chebyshev polynomials and Lattès examples.

Chebyshev polynomials. For any $n\in\mathbb{N}$ there exists a unique (real) polynomial p_n of degree n that satisfies the trigonometric identity $\cos n\theta = p_n(\cos\theta)$. It is called *Chebyshev polynomial.*

Lattès examples. Consider a one-dimensional complex torus, which is the quotient of $\mathbb{C}$ by a lattice. Consider arbitrary multiplication by a constant $\lambda\in\mathbb{C}$, $|\lambda|>1$, that maps the lattice to itself. It induces an endomorphism of the torus of degree greater than 1. The quotient of the torus by the central symmetry $z\mapsto -z$ is the Riemann sphere. The above endomorphism together with the quotient projection induce a rational transformation of the Riemann sphere called *Lattès example.*

Remark 1.13 Let f be either Chebyshev, or Lattès. Then it is critically finite. More precisely, the forward critical orbits eventually finish at repelling fixed points. The Julia set of a Chebyshev polynomial is the segment $[-1,1]$ of the real line, while that of a Lattès example is the whole Riemann sphere. Chebyshev and Lattès functions have branch-exceptional repelling fixed points, see the following definition.

Definition 1.14 [10] A periodic point of a rational function is called *branch-exceptional*, if any its nonperiodic backward orbit contains a critical point. In this case its periodic orbit is also called branch-exceptional.

Remark 1.15 (Lasse Rempe [10]). There exist quadratic rational functions with a branch-exceptional repelling fixed point that are neither Chebyshev, nor Lattès.

1.3 Background material 2: affine and hyperbolic dynamical laminations

The constructions presented here were introduced in [13]. We recall them briefly and send the reader to [13] for more details.

Recall that a *lamination* (by manifolds) is a "topological" foliation by manifolds, i.e., a topological space that is split as a disjoint union of manifolds (called *leaves*) of one and the same dimension so that each point of the ambient space admits a neighborhood (called "flow-box") such that each connected component (local leaf) of its intersection with each leaf is homeomorphic to a ball; the neighborhood itself is homeomorphic to the product of the ball and some (transverse) topological space under a homeomorphism transforming the local leaves to the fibers of the product.

Let $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ be a rational function. Set

$$\mathcal{N}_f=\{\hat z=(z_0,z_{-1},\dots)\mid z_{-j}\in\overline{\mathbb{C}},\ f(z_{-j-1})=z_{-j}\}.$$

This is a topological space equipped with the natural product topology and the projections

$$\pi_{-j} : \mathcal{N}_f \to \overline{\mathbb{C}},\ \hat{z} \mapsto z_{-j}.$$

The action of f on the Riemann sphere lifts naturally up to a homeomorphism

$$\hat{f} : \mathcal{N}_f \to \mathcal{N}_f,\ (z_0, z_{-1}, \dots) \mapsto (f(z_0), z_0, z_{-1}, \dots),\ f \circ \pi_{-j} = \pi_{-j} \circ \hat{f}.$$

First of all we recall the construction of the "regular leaf subspace" $\mathcal{R}_f \subset \mathcal{N}_f$, which is a union of Riemann surfaces that foliate $\mathcal{R}_f$ in a very turbulent way. Afterwards we take the subset $\mathcal{A}_f^n \subset \mathcal{R}_f$ of the leaves conformally-equivalent to $\mathbb{C}$. Then we refine the induced topology on $\mathcal{A}_f^n$ to make it a lamination (denoted $\mathcal{A}_f^l$) by complex lines with a continuous family of affine structures on them. Afterwards we take a completion $\mathcal{A}_f = \overline{\mathcal{A}_f^l}$ in the new topology. The space $\mathcal{A}_f$ is a lamination by affine Riemann surfaces (the new leaves added by the completion may have conical singularities). Then we discuss the three-dimensional extension of $\mathcal{A}_f$ up to a lamination $\mathcal{H}_f$ by hyperbolic manifolds (with singularities).

Let $\hat{z} \in \mathcal{N}_f$, $V = V(z_0) \subset \overline{\mathbb{C}}$ be a neighborhood of z_0. For any $j \geq 0$ set

$$V_{-j} = \text{the connected component of the preimage } f^{-j}(V) \text{ that contains } z_{-j}.$$

$$\text{Then } V_0 = V, \text{ and } f^j : V_{-j} \to V \text{ are ramified coverings.}$$

Definition 1.16 We say that a point $\hat{z} \in \mathcal{N}_f$ is *regular*, if there exists a disk V containing z_0 such that the above coverings $f^j : V_{-j} \to V$ have uniformly bounded degrees. Set

$$\mathcal{R}_f \subset \mathcal{N}_f \ \text{ the set of the regular points in } \mathcal{N}_f.$$

Example 1.17 Let $\hat{z} \in \mathcal{N}_f$ be a backward orbit such that there exists a $j \in \mathbb{N} \cup 0$ for which the point z_{-j} is disjoint from the ω- limit sets of the critical points. Then $\hat{z} \in \mathcal{R}_f$. If the mapping f is hyperbolic, then this is the case, if and only if $\hat{z}$ is not a (super) attracting periodic orbit. A mapping f is critically-nonrecurrent, if and only if

$$\mathcal{R}_f = \mathcal{N}_f \setminus \{\text{attracting and parabolic periodic orbits}\}, \text{ see [13]}.$$

Definition 1.18 Let $\hat{z} \in \mathcal{R}_f$, V, V_{-j} be as in the above definition. The *local leaf* $L(\hat{z}, V) \subset \mathcal{R}_f$ is the set of the points $\hat{z}' \in \mathcal{R}_f$ such that $z'_{-j} \in V_{-j}$ for all j (the local leaf is path-connected by definition). We say that the above local leaf is *univalent over* V, if the projection π_0 maps it bijectively onto V. The *global leaf* containing $\hat{z}$ (denoted $L(\hat{z})$) is the maximal path-connected subset in $\mathcal{R}_f$ containing $\hat{z}$.

Remark 1.19 Each leaf $L(\hat{z}) \subset \mathcal{R}_f$ carries a natural structure of Riemann surface so that the restrictions to the leaves of the above projections π_{-j} are meromorphic functions. A local leaf $L(\hat{z}, V) \subset L(\hat{z})$ (when well-defined) is the connected component containing $\hat{z}$ of the preimage $(\pi_0|_{L(\hat{z})})^{-1}(V) \subset L(\hat{z})$.

Remark 1.20 The above-defined objects $\mathcal{R}_f$, $\mathcal{R}_{f^n}$ corresponding to both f and any its forward iteration f^n, are naturally homeomorphic under the mapping that sends a backward orbit $\hat{z} \in \mathcal{N}_f$ to the backward orbit $(z_0, z_{-n}, z_{-2n}, \dots) \in \mathcal{N}_{f^n}$. The latter homeomorphism maps the leaves conformally onto the leaves.

Recall that given a rational function f and a $n \in \mathbb{N}$, a n- periodic connected component U of the Fatou set is called a *rotation domain: Siegel disk (Herman ring)*, if the restriction $f^n|_U$ is conformally conjugated to a rotation of disk (annulus).

We use the following

Lemma 1.21 (Shrinking Lemma, [13], p.86) *Let f be a rational mapping, $V \subset \overline{\mathbb{C}}$ be a domain, $V' \Subset V$ be a compact subset. Then for any sequence of single-valued inverse branches $f^{-n}: V \to \overline{\mathbb{C}}$ the diameters of the images $f^{-n}(V')$ tend to 0, as $n \to +\infty$, if ond only if f has no rotation domain that contains an infinite number of the above images.*

Definition 1.22 A noncompact Riemann surface is said to be *hyperbolic (parabolic),* if its universal covering is conformally equivalent to the unit disk (respectively, $\mathbb{C}$).

Remark 1.23 Parabolic leaves in $\mathcal{R}_f$ always exist (see the next two examples) and are simply connected; hence they are conformally equivalent to $\mathbb{C}$ ([13], lemma 3.3, p.27). If f is critically-nonrecurrent, then each leaf is parabolic ([13], proposition 4.5, p.36). On the other hand, there are rational mappings such that some leaves of $\mathcal{R}_f$ are hyperbolic (e.g., the mappings with rotation domains, see [13], p.27). Nontrivial examples of rational functions without rotation domains and with infinitely many hyperbolic leaves in $\mathcal{R}_f$, whose projections intersect the Julia set, were constructed in [10].

Example 1.24 Let $a \in \overline{\mathbb{C}}$ be a repelling fixed point of f, $\hat{a} = (a, a, \dots) \in \mathcal{N}_f$ be its fixed orbit. Then $\hat{a} \in \mathcal{R}_f$ and the leaf $L(\hat{a})$ is parabolic (it is $\hat{f}$- invariant and the quotient of $L(\hat{a}) \setminus \hat{a}$ by $\hat{f}$ is a torus). The linearizing coordinate w of f in a neighborhood of a lifts up to a conformal isomorphism $w \circ \pi_0 : L(\hat{a}) \to \mathbb{C}$. Analogously, the periodic orbit of a repelling periodic point is contained in a parabolic leaf (see Remark 1.20).

Example 1.25 Let f have a parabolic fixed point $a \in \overline{\mathbb{C}}$, $f'(a) = 1$, $\hat{a} = (a, a, \dots) \in \mathcal{N}_f$ be its fixed orbit. Then $\hat{a} \notin \mathcal{R}_f$. On the other hand, for each repelling ray (see Remark 1.12) there is a unique leaf in $\mathcal{R}_f$ (denoted L_a) consisting of the backward orbits that converge to a asymptotically along the chosen ray. The leaf L_a is parabolic: the Fatou coordinate w on the corresponding repelling sector lifts up to a conformal isomorphism $w \circ \pi_0 : L_a \to \mathbb{C}$. An analogous statement holds true in the case, when a is a parabolic periodic point and f is not necessarily tangent to the identity there.

Definition 1.26 The leaves from the two above examples are called respectively a *leaf associated to a repelling (parabolic) periodic point.*

Proposition 1.27 *A point $\hat{z} \in \mathcal{N}_f$ belongs to a leaf associated to a repelling (or parabolic) fixed point a, if and only if it is represented by a backward orbit converging to a (and distinct from its fixed orbit, if the latter is parabolic).*

The proposition follows from the Shrinking Lemma.

Corollary 1.28 *A leaf of $\mathcal{N}_f$ can be associated to at most one repelling or parabolic periodic point.*

Set

$$\mathcal{A}_f^n = \text{ the union of the parabolic leaves in } \mathcal{R}_f.$$

If f is hyperbolic, then $\mathcal{A}_f^n$ is a lamination with a global Cantor transverse section. In general, $\mathcal{A}_f^n$ is not a lamination in a good sense, since some ramified local leaves can accumulate to a univalent one in the product topology. The refined topology (defined in [13]) that makes it an orbifold lamination is recalled below. To do this, we use the following

Remark 1.29 Let $\hat{z} \in \mathcal{A}_f^n$. Fix a conformal isomorphism $\mathbb{C} \to L(\hat{z})$ that sends 0 to $\hat{z}$ (it is unique up to multiplication by nonzero complex constant in the source). For any $j \geq 0$ set

$$\phi_{-j,\hat{z}} = \pi_{-j}|_{L(\hat{z})} : \mathbb{C} = L(\hat{z}) \to \overline{\mathbb{C}}; \ \phi_{-j,\hat{z}} = f \circ \phi_{-j-1} \text{ for any } j, \ \phi_{-j,\hat{z}}(0) = z_{-j}. \tag{1.3}$$

This is a meromorphic function sequence, uniquely defined up to the $\mathbb{C}^*$- action on the source space $\mathbb{C}$ by multiplication by complex constants. Two points of $\mathcal{A}_f^n$ lie in one and the same leaf, if and only if the corresponding function sequences are obtained from each other by affine transformation of the source variable.

Let $\hat{\mathcal{K}}_f$ denote the space of the nonconstant meromorphic function sequences

$$\{\phi_{-j}(t)\}_{j\in\mathbb{N}\cup 0}, \ \phi_{-j} : \mathbb{C} \to \overline{\mathbb{C}}, \ \phi_{-j} = f \circ \phi_{-j-1} \text{ for all } j. \tag{1.4}$$

This is a subset of the infinite product of copies of the meromorphic function space; the latter space is equipped with the topology of uniform convergence on compact sets. The product topology induces a topology on the space $\hat{\mathcal{K}}_f$. The groups $Aff(\mathbb{C})$ (complex affine transformations of $\mathbb{C}$), $\mathbb{C}^* \subset Aff(\mathbb{C})$ and $S^1 = \{|z| = 1\} \subset \mathbb{C}^*$ act on the space $\hat{\mathcal{K}}_f$ by variable changes in the source. The stabilizer in $Aff(\mathbb{C})$ of a meromorphic function sequence from $\hat{\mathcal{K}}_f$ is a discrete group of Euclidean isometries. Set

$$\hat{\mathcal{K}}_f^a = \hat{\mathcal{K}}_f/\mathbb{C}^*, \ \hat{\mathcal{K}}_f^h = \hat{\mathcal{K}}_f/S^1. \tag{1.5}$$

The spaces $\hat{\mathcal{K}}_f^a$, $\hat{\mathcal{K}}_f^h$ are equipped with the quotient topologies induced from $\hat{\mathcal{K}}_f$. A *leaf* in $\hat{\mathcal{K}}_f^a$ ($\hat{\mathcal{K}}_f^h$) is the quotient projection of an $Aff(\mathbb{C})$- orbit in $\hat{\mathcal{K}}_f$.

Remark 1.30 A leaf in $\hat{\mathcal{K}}_f^a$ ($\hat{\mathcal{K}}_f^h$) is naturally identified with the quotient of $Aff(\mathbb{C})/\mathbb{C}^* = \mathbb{C}$ (respectively, $Aff(\mathbb{C})/S^1 = \mathbb{H}^3$) by the left action of a discrete subgroup of Euclidean isometries in $Aff(\mathbb{C})$. This equips the leaves with natural *affine (hyperbolic) structures* that *vary continuously* on the ambient space. Each leaf is thus an affine (hyperbolic) orbifold. There is a natural inclusion

$$\mathcal{A}_f^n \to \hat{\mathcal{K}}_f^a : \ \hat{z} \mapsto \{\phi_{-j,\hat{z}}\}_{j=0}^{+\infty}/\mathbb{C}^*, \text{ see (1.3)}. \tag{1.6}$$

The action of $\mathbb{C}^* = S^1 \times \mathbb{R}_+^*$ on $\hat{\mathcal{K}}_f$ induces an action of $\mathbb{R}_+^*$ on $\hat{\mathcal{K}}_f^h$. The quotient of the latter action is $\hat{\mathcal{K}}_f^a = \hat{\mathcal{K}}_f/\mathbb{C}^* = \hat{\mathcal{K}}_f^h/\mathbb{R}_+^*$. The corresponding quotient projection will be denoted

$$\pi_{hyp} : \hat{\mathcal{K}}_f^h \to \hat{\mathcal{K}}_f^a. \tag{1.7}$$

The projection π_{hyp} maps each hyperbolic leaf of $\hat{\mathcal{K}}_f^h$ onto an affine leaf of $\hat{\mathcal{K}}_f^a$ that is canonically identified with the boundary of the hyperbolic leaf. Conversely, the preimage of an affine leaf is a hyperbolic leaf.

Definition 1.31 The topological subspace $\mathcal{A}_f^l \subset \hat{\mathcal{K}}_f^a$ is the image of the space $\mathcal{A}_f^n$ under the inclusion (1.6), or equivalently, the space $\mathcal{A}_f^n$ equipped with the topology induced from $\hat{\mathcal{K}}_f^a$. The space $\mathcal{A}_f$ is the closure of $\mathcal{A}_f^l$ in the space $\hat{\mathcal{K}}_f^a$. We set

$$\mathcal{H}_f^l = \pi_{hyp}^{-1}(\mathcal{A}_f^l) \subset \hat{\mathcal{K}}_f^h, \ \mathcal{H}_f = \pi_{hyp}^{-1}(\mathcal{A}_f) = \overline{\mathcal{H}_f^l} \subset \hat{\mathcal{K}}_f^h. \tag{1.8}$$

The space $\mathcal{A}_f$ ($\mathcal{H}_f$) is called the *affine (hyperbolic) orbifold lamination associated to* f. For any point $\hat{z} \in \mathcal{A}_f$ the affine leaf through $\hat{z}$ in $\mathcal{A}_f$ will be denoted by $L(\hat{z})$, and set

$$H(\hat{z}) = \pi_{hyp}^{-1}(L(\hat{z})) \subset \mathcal{H}_f: \text{ the corresponding hyperbolic leaf.}$$

Remark 1.32 In general, the topology of the space $\mathcal{A}_f^l$ is stronger than that of $\mathcal{A}_f^n$. The spaces $\mathcal{A}_f^l$, $\mathcal{A}_f$, $\mathcal{H}_f^l$, $\mathcal{H}_f$ consist of entire leaves. Each leaf in $\mathcal{A}_f^l$ ($\mathcal{H}_f^l$) is affine-isomorphic (isometric) to $\mathbb{C}$ (respectively, $\mathbb{H}^3$). Other leaves may contain conical singularities.

There is a natural projection

$$p: \hat{\mathcal{K}}_f^a \to \mathcal{A}_f^n \tag{1.9}$$

induced by the mapping $\hat{\mathcal{K}}_f \to \mathcal{A}_f^n$ that sends each sequence (1.4) of functions to the sequence of their values at 0. The latter value sequence is always a regular backward orbit of f, and it lies in a parabolic leaf of $\mathcal{R}_f$. The regularity follows from definition. The parabolicity follows from Picard's theorem. The composition of p with the natural inclusion $\mathcal{A}_f^n \to \hat{\mathcal{K}}_f^a$ is the identical mapping $\mathcal{A}_f^n \to \mathcal{A}_f^n$. The projection

$$\mathcal{A}_f \to \overline{\mathbb{C}} \text{ induced by } \pi_0, \ (\phi_{-j})_{j \in \mathbb{N} \cup 0} \mapsto \phi_0(0), \text{ will be also denoted by } \pi_0. \tag{1.10}$$

The rational mapping $f: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ lifts up to the leafwise homeomorphism

$$\hat{f}: \hat{\mathcal{K}}_f \to \hat{\mathcal{K}}_f, \ \hat{f}: (\phi_0, \phi_{-1}, \dots) \mapsto (f \circ \phi_0, \phi_0, \phi_{-1}, \dots), \text{ which induces homeomorphisms}$$

$$\hat{f}: \mathcal{A}_f \to \mathcal{A}_f \text{ leafwise affine and } \hat{f}: \mathcal{H}_f \to \mathcal{H}_f \text{ leafwise isometric; } \hat{f}(\mathcal{A}_f^l) = \mathcal{A}_f^l.$$

These homeomorphisms form a commutative diagram with the projection π_{hyp}. The latter homeomorphism acts properly discontinuously on $\mathcal{H}_f$, and its quotient

$\mathcal{H}_f/\hat{f}$ is Hausdorff and called the *quotient hyperbolic lamination associated to* f.

Proposition 1.33 *([13], proposition 7.5, p.62) A sequence of points $\hat{z}^m \in \mathcal{A}_f^l$ converges to a point $\hat{z} \in \mathcal{A}_f^l$, as $m \to \infty$, if and only if*

- *$\pi_{-j}(\hat{z}^m) \to \pi_{-j}(\hat{z})$ for any j,*
- *for every $N \in \mathbb{N}$, each connected domain $V \subset \overline{\mathbb{C}}$ and each its subdomain U such that $\overline{U} \subset V$, $\pi_{-N}(\hat{z}) \in U$, if the local leaf $L(\hat{f}^{-N}(\hat{z}), V)$ is univalent over V, then the local leaf $L(\hat{f}^{-N}(\hat{z}^m), U)$ is univalent over U, whenever m is large enough.*

Remark 1.34 The analogous criterion holds true for convergence of a sequence of points in $\mathcal{A}_f$ to a point in $\mathcal{A}_f^l$ with the following definition of local leaf in $\mathcal{A}_f$.

Definition 1.35 Let f be a rational mapping, $\mathcal{A}_f$ be the corresponding affine lamination, $L \subset \mathcal{A}_f$ be a leaf, $\hat{z} \in L$, $V \subset \overline{\mathbb{C}}$ be a domain containing its projection $\pi_0(\hat{z})$. The *local leaf* $L(\hat{z}, V)$ is the connected component containing $\hat{z}$ of the projection preimage $\pi_0^{-1}(V) \cap L$. A local leaf is called *univalent over* V, if it contains no singular points and is bijectively projected onto V.

Corollary 1.36 *Let a be a repelling fixed point of f, $\hat{a} \in \mathcal{A}_f^l$ be its fixed orbit. Let $V = V(a) \subset \overline{\mathbb{C}}$ be a neighborhood of a, $\{\hat{b}^m\}_{m\in\mathbb{N}}$ be a sequence of points in $\mathcal{A}_f$ such that $\pi_0(\hat{b}^m) = a$ and the local leaves $L(\hat{b}^m, V)$ are univalent over V (see the above definition). Then $\hat{f}^m(\hat{b}^m) \to \hat{a}$, as $m \to +\infty$.*

Proof For simplicity we assume that $\hat{b}^m \in \mathcal{A}_f^l$ (the opposite case is treated analogously with minor modifications, taking into account Remark 1.34). By Proposition 1.33, for the proof of the convergence $\hat{f}^m(\hat{b}^m) \to \hat{a}$ it suffices to prove the statements of the proposition for $\hat{z} = \hat{a}$ and $\hat{z}^m = \hat{f}^m(\hat{b}^m)$. Its first statement, which says that $\pi_{-j}(\hat{f}^m(\hat{b}^m)) \to a$, as $m \to \infty$, for all j, follows from construction: $\pi_{-j}(\hat{f}^m(\hat{b}^m)) = \pi_0(\hat{f}^{m-j}(\hat{b}^m)) = a$, whenever $m \geq j$. Let us prove its second statement on univalence. Without loss of generality we consider that the local leaf $L(\hat{a}, V)$ is univalent over V (one can achieve this by shrinking V). By definition, $\hat{f}^{-N}(\hat{a}) = \hat{a}$ for any $N \in \mathbb{N}$. Fix an arbitrary domain $W \subset \overline{\mathbb{C}}$, $a \in W$, such that the local leaf $L(\hat{a}, W)$ is univalent over W and a subdomain U, $\overline{U} \subset W$, $a \in U$. It suffices to show that for any $N \in \mathbb{N}$ the local leaf $L(\hat{f}^{m-N}(\hat{b}^m), U)$ is univalent over U, whenever m is large enough. To do this, consider the inverse branches f^{-s} that fix a: they are analytic on W and tend to a uniformly on $\overline{U}$, as $s \to +\infty$ (by definition and the Shrinking Lemma). Hence, any of these branches maps W diffeomorphically onto its image. Therefore, there exists a $k \in \mathbb{N}$, $k \geq N$, such that $f^{N-m}(\overline{U}) \subset V$ for any $m \geq k$. Fix an arbitrary $m \geq k \geq N$. Then $f^{N-m} : U \to f^{N-m}(U)$ is a conformal diffeomorphism. The local leaf $L(\hat{b}^m, f^{N-m}(U))$ is univalent over $f^{N-m}(U)$. Indeed, it is contained in the local leaf $L(\hat{b}^m, V)$, which is univalent over V by the conditions of the corollary. Together with the above statement, this implies that the local leaf $L(\hat{f}^{m-N}(\hat{b}^m), U)$ is univalent over U. This proves the corollary. □

Definition 1.37 A leaf of $\mathcal{A}_f$ is *associated to a repelling (or parabolic) periodic point* if it is contained in $\mathcal{A}_f^l$ and coincides with a leaf of $\mathcal{A}_f^n$ that is associated to the above point (see Definition 1.26). In this case we also say that the corresponding leaves of $\mathcal{H}_f$ and $\mathcal{H}_f/\hat{f}$ are associated to this point.

Proposition 1.38 *[10, 13] The laminations $\mathcal{A}_f$ and $\mathcal{H}_f$ are minimal (i.e., each leaf is dense), if and only if the function f does not have branch-exceptional repelling periodic orbits (see Definition 1.14). If f has branch-exceptional repelling periodic orbits, then each of the above laminations has a finite number of isolated leaves (at most four; all of them are associated to the latter periodic orbits) and becomes minimal after removing the isolated leaves. The isolated leaves accumulate to their (minimal) complement, $\mathcal{A}_f'$ or $\mathcal{H}_f'$:*

$$\mathcal{A}_f' = \mathcal{A}_f \setminus (\textit{the isolated leaves}); \ \mathcal{H}_f' = \pi_{hyp}^{-1}(\mathcal{A}_f') = \mathcal{H}_f \setminus (\textit{the isolated leaves}). \tag{1.11}$$

One has $\mathcal{H}_f' = \mathcal{H}_f$, if and only if f does not have branch-exceptional repelling periodic orbits.

1.4 Background material 3: horospheres; metric properties and basic cocycle

The horospheres in the hyperbolic 3- space with a marked point "infinity" at the boundary (and in the leaves of the hyperbolic laminations) were defined in Subsection 1.1. We use their following well-known equivalent definition. Consider the projection $\pi : \mathbb{H}^3 \to L = \partial\mathbb{H}^3 \setminus \infty$ to the boundary plane along the geodesics issued from the infinity. In the model of half-space this is the Euclidean orthogonal projection to the boundary plane. It coincides with the natural projection $\mathbb{H}^3 = Aff(\mathbb{C})/S^1 \to \mathbb{C} = Aff(\mathbb{C})/\mathbb{C}^*$, and its latter description equips the boundary with a natural complex affine structure: $L = \mathbb{C}$. The boundary admits a Euclidean affine metric (uniquely defined up to multiplication by constant).

Everywhere below whenever we consider a Riemannian metric on a surface, we treat it as a length element, not as a quadratic form. If we say "two metrics are proportional", then by definition, the proportionality coefficient is the ratio of the corresponding length elements.

Consider a global section of the projection $\pi : \mathbb{H}^3 \to L$: a surface in $\mathbb{H}^3$ that is 1-to-1 projected to L. It carries two metrics: the restriction to it of the hyperbolic metric of the ambient space $\mathbb{H}^3$; the pullback of the Euclidean metric of L under the projection.

Definition 1.39 The above section is a *horosphere*, if its latter (Euclidean) metric is obtained from the former one (the restricted hyperbolic metric) by multiplication by a constant factor. The *height* of a horosphere (with respect to the chosen Euclidean metric on L) is the logarithm of the latter constant factor. The *height of a given point* in the hyperbolic space is the height of the horosphere that contains this point.

Remark 1.40 The height is a real-valued analytic function $\mathbb{H}^3 \to \mathbb{R}$. In the upper half-space model the horospheres are horizontal planes. Their heights (with respect to the standard Euclidean metric on $\partial\mathbb{H}^3 \setminus \infty$) are equal to the logarithms of their Euclidean heights in the ambient 3- space. The isometric liftings to $\mathbb{H}^3$ of the affine mappings $z \mapsto \lambda z + b$ of the boundary $\mathbb{C} = \partial\mathbb{H}^3 \setminus \infty$ transform the horospheres to the horospheres so that the height of the image equals $\ln|\lambda|$ plus the height of the preimage. The horospheres foliate $\mathbb{H}^3$. A quotient of $\mathbb{H}^3$ by a discrete group of isometries fixing ∞ (e.g., a leaf of $\mathcal{H}_f$ or $\mathcal{H}_f/\hat{f}$) carries the quotient foliation by horospheres.

Now we discuss metric properties of the horospheres in the hyperbolic laminations. Let $\mathcal{A}_f$, $\mathcal{H}_f$ be respectively the affine and the hyperbolic laminations associated to a rational function f.

Proposition 1.41 *The horospheres form a lamination of any of the spaces $\mathcal{H}_f$, $\mathcal{H}_f/\hat{f}$ by affine surfaces (orbifolds that may have conical singularities). In particular, the closure of a union of horospheres is a union of horospheres.*

The proposition follows from the continuity of the hyperbolic metrics (Remark 1.30).

Let $L \subset \mathcal{A}_f$ be a leaf, $\hat{z} \in L$ be a nonsingular point such that the restricted projection $\pi_0|_L$ has nonzero derivative at $\hat{z}$. Fix a Hermitian metric on the tangent line to $\overline{\mathbb{C}}$ at $\pi_0(\hat{z})$. Its projection pullback to the tangent line $T_{\hat{z}}L$ extends (in unique way) up to a Euclidean affine metric on the whole leaf L. Let H be the corresponding leaf in $\mathcal{H}_f$. We set

$$\beta_{\hat{z}} : H \to \mathbb{R} \text{ the height with respect to the latter metric on } L, \text{ see Definition 1.39,} \quad (1.12)$$

$$\alpha = (\hat{z}, h) \in H \text{ the point such that } \pi_{hyp}(\alpha) = \hat{z} \text{ and } \beta_{\hat{z}}(\alpha) = h,$$

then we say that the point α is *situated over* $\hat{z}$ at height h,

$$S_{\hat{z},h} \subset H \text{ the horosphere such that } \beta_{\hat{z}}|_{S_{\hat{z},h}} \equiv h. \tag{1.13}$$

Proposition 1.42 *A sequence of points $(\hat{z}^k, h_k) \in \mathcal{H}_f$ converges to a point $(\hat{z}, h) \in \mathcal{H}_f$, if and only if $\hat{z}^k \to \hat{z}$ in $\mathcal{A}_f$ and $h_k \to h$.*

The proposition follows from the continuity of the hyperbolic metrics of the leaves.

When we extend the horospheres along loops in $\overline{\mathbb{C}}$, their heights may change. The monodromy of the heights is described by basic cocycle. Let us recall its definition.

Definition 1.43 Let $L \subset \mathcal{A}_f$ be a leaf, $\hat{z}, \hat{z}' \in L$ be a pair of nonsingular points projected to one and the same $z = \pi_0(\hat{z}) = \pi_0(\hat{z}') \in \overline{\mathbb{C}}$ so that the restricted projection $\pi_0|_L$ has nonzero derivative at both points $\hat{z}$ and $\hat{z}'$. Let $H = H(\hat{z}) \subset \mathcal{H}_f$ be the corresponding hyperbolic leaf. Fix a Hermitian metric on $T_z\overline{\mathbb{C}}$. Let $\beta_{\hat{z}}, \beta_{\hat{z}'} : H \to \mathbb{R}$ be the corresponding heights defined in (1.12). The *basic cocycle* is the difference

$$\beta(\hat{z}, \hat{z}') = \beta_{\hat{z}'} - \beta_{\hat{z}}.$$

Remark 1.44 In the conditions of the above definition the basic cocycle is a well-defined constant and depends only on $\hat{z}$ and $\hat{z}'$ (it is independent on the choice of metric). One has

$$\beta(\hat{z}, \hat{z}) = 0, \ \beta(\hat{z}, \hat{z}') = -\beta(\hat{z}', \hat{z}).$$

Each horosphere $S_{\hat{z},h} \subset H(\hat{z})$ coincides with the horosphere $S_{\hat{z}',h+\beta(\hat{z},\hat{z}')}$. For any triple of nonsingular points $\hat{z}, \hat{z}', \hat{z}'' \in \mathcal{A}_f$ lying in one and the same leaf L and projected by $\pi_0|_L$ to one and the same point $z \in \overline{\mathbb{C}}$ with nonzero derivatives one has

$$\beta(\hat{z}', \hat{z}'') = \beta(\hat{z}, \hat{z}'') - \beta(\hat{z}, \hat{z}') \text{ (the cocycle identity).} \tag{1.14}$$

The basic cocycle is $\hat{f}$- invariant:

$$\beta(\hat{z}, \hat{z}') = \beta(\hat{f}^n(\hat{z}), \hat{f}^n(\hat{z}')) \text{ for any } n \in \mathbb{N}. \tag{1.15}$$

The next proposition is well-known and follows immediately from definition.

Proposition 1.45 *Let $L \subset \mathcal{A}_f$ be a leaf, $\hat{c}, \hat{c}' \in L$, $\pi_0(\hat{c}) = \pi_0(\hat{c}') = c$. Let $V \subset \overline{\mathbb{C}}$ be a neighborhood of c such that the local leaves $L(\hat{c}, V), L(\hat{c}', V) \subset L$ are univalent over V (see Definition 1.35). Consider the conformal isomorphism*

$$\psi_{\hat{c},\hat{c}'} = (\pi_0|_{L(\hat{c}',V)})^{-1} \circ \pi_0|_{L(\hat{c},V)} : L(\hat{c}, V) \to L(\hat{c}', V). \tag{1.16}$$

Let us fix a Euclidean affine metric on the leaf L, which contains the above local leaves. Consider the module $|\psi'_{\hat{c},\hat{c}'}|$ of derivative in the chosen Euclidean metric. Then for any $\hat{z} \in L(\hat{c}, V)$, set $\hat{z}' = \psi_{\hat{c},\hat{c}'}(\hat{z})$, one has

$$\beta(\hat{z}, \hat{z}') = -\ln |\psi'_{\hat{c},\hat{c}'}(\hat{z})|. \tag{1.17}$$

Corollary 1.46 *Let L, $\hat{c}$, $\hat{c}'$, V be as in the above proposition. For any $z \in V$ set*

$$\hat{z} = \pi_0^{-1}(z) \cap L(\hat{c}, V), \ \hat{z}' = \pi_0^{-1}(z) \cap L(\hat{c}', V). \textit{ The function}$$

$$\beta_{\hat{c},\hat{c}'}(z) = \beta(\hat{z}, \hat{z}') \tag{1.18}$$

is harmonic on V, and hence, real-analytic.

1.5 Main results

First let us recall the following

Theorem 1.47 *([11], p.62) The affine lamination $\mathcal{A}_f$ associated to a rational function f admits a continuous family of Euclidean affine metrics on the non-isolated leaves, if and only if f is conformally-conjugated to a function from the list (1.1). In the latter case there exists a unique (up to multiplication by constant) conformal Euclidean metric on $\overline{\mathbb{C}}$ with isolated singularities whose pullback to the non-isolated leaves under the projection $\pi_0 : \mathcal{A}'_f \to \overline{\mathbb{C}}$ yields the above Euclidean metric family on the non-isolated leaves.*

Corollary 1.48 *Let f be a rational function from (1.1). Then each horosphere in any non-isolated leaf of its quotient hyperbolic lamination $\mathcal{H}_f/\hat{f}$ is nowhere dense in $\mathcal{H}_f/\hat{f}$.*

Proof Let S be an arbitrary horosphere in a non-isolated leaf of $\mathcal{H}_f$. For the proof of the corollary it suffices to show that the images of S under forward and backward iterations of $\hat{f}$ are nowhere dense. Let g denote the singular Euclidean metric on $\overline{\mathbb{C}}$ from the above theorem. We measure the heights of the horospheres with respect to the metric g lifted to the leaves of $\mathcal{A}'_f$ under the projection π_0. The heights of S over all the points of the corresponding leaf of the affine lamination are all the same (by definition and Theorem 1.47). The mapping f has a constant module of derivative in the metric g, since $\hat{f}$ is leafwise affine. The heights of the iterated images of S form an arithmetic progression with step equal to the logarithm of the latter module of derivative. This progression is a discrete set of real numbers. Hence, the union of the images of S is nowhere dense. This proves the corollary. □

Recall that $\mathcal{H}'_f$ denotes the union of the non-isolated leaves in $\mathcal{H}_f$. It is $\hat{f}$- invariant, and its quotient $\mathcal{H}'_f/\hat{f}$ is the union of the non-isolated leaves in $\mathcal{H}_f/\hat{f}$.

Theorem 1.49 *Let f be a rational function that does not belong to the list (1.1). Let $\mathcal{H}_f/\hat{f}$ ($\mathcal{H}'_f/\hat{f}$) be the corresponding quotient hyperbolic lamination (with deleted isolated leaves, see (1.11)). Let $H/\hat{f} \subset \mathcal{H}'_f/\hat{f}$ be a non-isolated leaf associated to a repelling periodic point of f (see Definition 1.37). Then each horosphere in $H/\hat{f}$ is dense in $\mathcal{H}'_f/\hat{f}$.*

Remark 1.50 The latter repelling periodic point is not branch-exceptional (Definition 1.14, Proposition 1.38 and Corollary 1.28).

Theorem 1.49 is the principal result of the paper. It is proved in Section 2. As it is shown below, it implies density of all the horospheres in $\mathcal{H}'_f/\hat{f}$ in the critically-nonrecurrent nonparabolic case and density of "almost" all the horospheres in the general critically-nonrecurrent case, provided that $f \notin (1.1)$.

Theorem 1.51 *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a critically-nonrecurrent rational function without parabolic periodic points (e.g., a hyperbolic one) that does not belong to the list (1.1). Then each horosphere in $\mathcal{H}_f/\hat{f}$ accumulates to $\mathcal{H}'_f/\hat{f}$.*

Theorem 1.52 *Let f be a critically-nonrecurrent rational function that does not belong to the list (1.1). Let $H \subset \mathcal{H}_f$ be an arbitrary hyperbolic leaf, $L = \pi_{hyp}(H) \subset \mathcal{A}_f$ be the corresponding affine leaf. Let the projection $p(L) \subset \mathcal{A}^n_f$ do not lie in a leaf associated to*

a parabolic periodic point of f. Let $H/\hat{f} \subset \mathcal{H}_f/\hat{f}$ be the corresponding leaf of the quotient lamination. Then each horosphere in $H/\hat{f}$ accumulates to $\mathcal{H}'_f/\hat{f}$.

Theorem 1.51 follows immediately from Theorem 1.52. Below we deduce Theorem 1.52 from Theorem 1.49 and the following theorem, which will be proved in Section 3.

Theorem 1.53 *Let the conditions of Theorem 1.52 hold, but now f is not necessarily excluded from the list (1.1). Then each horosphere in $H/\hat{f}$ accumulates to some horosphere in a leaf in $\mathcal{H}_f/\hat{f}$ associated to an appropriate repelling periodic point that is not branch-exceptional.*

Proof of Theorem 1.52. Each horosphere in $H/\hat{f}$ accumulates to some horosphere in a leaf in $\mathcal{H}_f/\hat{f}$ corresponding to some repelling periodic point that is not branch-exceptional (Theorem 1.53). The latter horosphere accumulates to $\mathcal{H}'_f/\hat{f}$ (Theorem 1.49). Hence, so does the former horosphere. This proves Theorems 1.52 and 1.51. $\square$

The following theorem proves the converse for the horospheres in the leaves associated to parabolic periodic points, without the critical nonrecurrence assumption.

Theorem 1.54 *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be an arbitrary rational function with a parabolic periodic point a. Let $H_a \subset \mathcal{H}_f$ be a leaf associated to it, $H_a/\hat{f} \subset \mathcal{H}_f/\hat{f}$ be the corresponding leaf of the quotient hyperbolic lamination. Each horosphere in H_a ($H_a/\hat{f}$) is closed in $\mathcal{H}_f$ (respectively, $\mathcal{H}_f/\hat{f}$) and does not accumulate to itself.*

Theorem 1.54 will be proved in Section 4.

The next theorem shows that Theorems 1.49-1.52 are false when formulated for the non-factorized lamination $\mathcal{H}_f$. It deals with the quadratic polynomial family

$$f_\varepsilon(x) = x^2 + \varepsilon, \ \varepsilon < \frac{1}{4}. \text{ Set}$$

$$a(\varepsilon) = \frac{1+\sqrt{1-4\varepsilon}}{2} : \text{ this is the maximal real fixed point of } f_\varepsilon. \tag{1.19}$$

Remark 1.55 A polynomial f_ε belongs to the list (1.1), if and only if either $\varepsilon = 0$, or $\varepsilon = -2$. In the latter case f_ε is conformally conjugated to a Chebyshev polynomial.

Proposition 1.56 *The fixed point $a(\varepsilon)$ is repelling. It is branch-exceptional, if and only if $\varepsilon = -2$.*

Proof Recall that $\varepsilon < \frac{1}{4}$. The fixed point equation $x^2 - x + \varepsilon = 0$ has two distinct solutions, one of them is $a(\varepsilon)$. Therefore, $f'_\varepsilon(a(\varepsilon)) \neq 1$. One has $a(\varepsilon) > 0$, by (1.19). Thus, $a(\varepsilon)$ is not a parabolic fixed point: $1 \neq f'(a(\varepsilon)) = 2a(\varepsilon) > 0$. On the other hand, the polynomial f_ε has no fixed points on the right from $a(\varepsilon)$ by definition. It also has no critical points there, since $f'(x) = 2x > 2a(\varepsilon) > 0$ for any $x > a(\varepsilon)$. This implies that the orbit of each point $x > a(\varepsilon)$ converges to infinity, since this is true whenever x is large enough. Hence, $a(\varepsilon)$ is repelling. The quadratic mapping f_ε has unique critical point 0, and $f_\varepsilon^{-1}(a(\varepsilon)) = \{\pm a(\varepsilon)\}$. Hence, the fixed point $a(\varepsilon)$ is branch-exceptional, if and only if $-a(\varepsilon)$ is the critical value, i.e., $-a(\varepsilon) = \varepsilon$. The latter equation has the unique real solution $\varepsilon = -2$. $\square$

Theorem 1.57 *Let $\varepsilon \in (-\infty, \frac{1}{4}) \setminus \{-2, 0\}$, f_ε, $a(\varepsilon)$ be as above, $\hat{a} \in \mathcal{A}_{f_\varepsilon}$ be the fixed orbit of $a(\varepsilon)$, $H(\hat{a}) \subset \mathcal{H}_{f_\varepsilon}$ be the corresponding leaf. Then each horosphere in $H(\hat{a})$ is nowhere dense in $\mathcal{H}_{f_\varepsilon}$. More precisely, if $\pm\varepsilon > 0$, then there exists a countable closed (and thus, nowhere dense) additive semigroup $B_\varepsilon \subset \mathbb{R}_\pm$ such that for any $h \in \mathbb{R}$ the accumulation set of the horosphere $S_{\hat{a},h} \subset H(\hat{a})$ is the horosphere union*

$$\cup_{\beta \in B_\varepsilon} S_{\hat{a},h+\beta} \subset H(\hat{a}).$$

Addendum. *There exists an open set $W \subset \mathbb{C}$ containing the interval union $(-\infty, 0) \cup (0, \frac{1}{4})$ such that the statements of Theorem 1.57 hold true for all $\varepsilon \in W \setminus \{-2\}$, with $\pm\varepsilon > 0$ replaced by $\pm \operatorname{Re} \varepsilon > 0$.*

Theorem 1.57 and its addendum will be proved in Section 5.

2 Density of the horospheres over repellers. Proof of Theorem 1.49

2.1 The plan of the proof of Theorem 1.49

Let $a \in \overline{\mathbb{C}}$ be some repelling periodic point of f that is not branch-exceptional (see Remark 1.50). Let $\hat{a} \in \mathcal{A}_f$ be its periodic backward orbit, $L(\hat{a})$, $H(\hat{a})$ be the respectively the corresponding leaves of the laminations $\mathcal{A}_f$ and $\mathcal{H}_f$. Then the leaves $L(\hat{a})$, $H(\hat{a})$ are contained respectively in $\mathcal{A}'_f$ and $\mathcal{H}'_f$, and they are dense there (Proposition 1.38 and Corollary 1.28). We fix a horosphere $S \subset H(\hat{a})$, set

$$\mathcal{S} = \cup_{m \in \mathbb{Z}} \hat{f}^m(S), \text{ and show that the closure of } \mathcal{S} \text{ in } \mathcal{H}_f \text{ contains } H(\hat{a}). \tag{2.1}$$

Then $\mathcal{S}$ is dense in $\mathcal{H}'_f$, as is $H(\hat{a})$. This proves Theorem 1.49.

It suffices to prove (2.1) with $S = S_{\hat{a},0}$. This implies the same statement for any other horosphere $S_{\hat{a},h}$, since for any $\hat{y} \in L(\hat{a})$, $h \in \mathbb{R}$ and $m \in \mathbb{Z}$

$$(\text{the height of } \hat{f}^m(S_{\hat{a},h}) \text{ over } \hat{y}) \ = \ h + (\text{the height of } \hat{f}^m(S_{\hat{a},0}) \text{ over } \hat{y}). \tag{2.2}$$

Without loss of generality everywhere below we assume that the point a is fixed: $f(a) = a$. One can achieve this by replacing f by its iterate: statement (2.1) for an iterate, which we will prove, is stronger than that for f. Then both leaves $L(\hat{a})$ and $H(\hat{a})$ are fixed by $\hat{f}$, which acts on $L(\hat{a})$ by (complex) homothety centered at $\hat{a}$ with coefficient $f'(a)$. Set

$$\Pi_a = \{\hat{y} \in L(\hat{a}) \setminus \hat{a} \mid \pi_0(\hat{y}) = a, \ (\pi_0|_{L(\hat{a})})'(\hat{y}) \neq 0\}. \tag{2.3}$$

Each horosphere $S \subset H(\hat{a})$ is mapped by $\hat{f}$ to a horosphere in the same leaf $H(\hat{a})$ so that

$$\hat{f}^m(S_{\hat{a},0}) = S_{\hat{a},m \ln|f'(a)|}, \ \hat{f}(S_{\hat{y},h}) = S_{\hat{f}(\hat{y}),h+\ln|f'(a)|} \text{ for any } m \in \mathbb{Z}, \ h \in \mathbb{R} \text{ and } \hat{y} \in \Pi_a. \tag{2.4}$$

The monodromies of the horospheres (when defined) along loops based at a add appropriate basic cocycles to the heights (see Definition 1.43) so that for any $\hat{y} \in \Pi_a$, $h \in \mathbb{R}$, $m \in \mathbb{Z}$

$$S_{\hat{a},h} = S_{\hat{y},h+\beta(\hat{a},\hat{y})}, \text{ thus, } \hat{f}^m(S_{\hat{a},0}) = S_{\hat{y},h_{\hat{y},m}}, \ h_{\hat{y},m} = \beta(\hat{a},\hat{y}) + m \ln|f'(a)|. \tag{2.5}$$

The main part of the proof of Theorem 1.49 is the next lemma, which implies that the above height values $h_{\hat{y},m}$ are dense in $\mathbb{R}$. Theorem 1.49 is then deduced from it by elementary topological arguments (using Corollary 1.36), which are presented at the end of the subsection.

Proposition 2.1 *Let f be a rational function, a be some its repelling fixed point that is not branch-exceptional. Then the set Π_a from (2.3) is nonempty.*

Proof The leaf $L(\hat{a})$ is contained in $\mathcal{A}'_f$, is dense there, and in particular, accumulates to itself (see the beginning of the subsection). Therefore, there exist a neighborhood $U = U(a)$ and a sequence of points $\hat{a}^n \in L(\hat{a})$ converging to $\hat{a}$ such that all the local leaves $L(\hat{a}, U)$, $L(\hat{a}^n, U)$ are univalent and distinct (see Proposition 1.33). Then without loss of generality we consider that $a_0^n = a$. By construction, $\hat{a}^n \neq \hat{a}$ for infinite number of indices n, and hence, $\hat{a}^n \in \Pi_a$ (the above univalence statement). □

Lemma 2.2 *Let f be a rational function that does not belong to the list (1.1), a be some its repelling fixed point that is not branch-exceptional. Let Π_a be as in (2.3). The set*

$$\mathcal{B}f = \{\beta(\hat{a}, \hat{y}) + m \ln |f'(a)| \mid \hat{y} \in \Pi_a, \ m \in \mathbb{Z}\} \tag{2.6}$$

is dense in $\mathbb{R}$.

Everywhere below for any $z \in \overline{\mathbb{C}}$ (with a chosen local holomorphic chart in its neighborhood, the latter being equipped with the standard Euclidean metric) and $\delta > 0$ we set

$$D_\delta(z) = \text{ the } \delta- \text{ disk centered at } z, \ D_\delta = D_\delta(0).$$

The proof of Lemma 2.2 modulo technical details is given below. The details of the proof take the most part of the section. In its proof we use the following properties of the points from Π_a and basic cocycles.

Proposition 2.3 *Let f be a rational function, $a \in \overline{\mathbb{C}}$ be some its repelling fixed point that is not branch-exceptional. Let Π_a be as in (2.3), $\hat{y}, \hat{c} \in \Pi_a$. Let $\delta > 0$ be such that the local leaves $L(\hat{a}, D_\delta(a))$, $L(\hat{y}, D_\delta(a))$, $L(\hat{c}, D_\delta(a))$ are univalent over $D_\delta(a)$, and moreover, the inverse branch f^{-1} that fixes a extends up to a univalent holomorphic function $D_\delta(a) \to D_\delta(a)$ (whose orbits in $D_\delta(a)$ thus converge to a). Let $j \in \mathbb{N}$ be such that $y_{-k} \in D_\delta(a)$ for any $k \geq j$ (see Proposition 1.27). Let*

$$\hat{\zeta} \in L(\hat{c}, D_\delta(a)), \ \pi_0(\hat{\zeta}) = y_{-j}, \ \hat{w} = \hat{f}^j(\hat{\zeta}), \ \beta_{\hat{a},\hat{c}} \text{ be the function from (1.18). Then}$$

$$\hat{w} \in \Pi_a \text{ and } \beta(\hat{a}, \hat{w}) = \beta(\hat{a}, \hat{y}) + \beta_{\hat{a},\hat{c}}(y_{-j}). \tag{2.7}$$

Proof One has $\hat{w} \in \Pi_a$. Indeed, $\pi_0(\hat{w}) = f^j(y_{-j}) = a$. No w_{-k} is a critical point of f. For $k > j$ this follows from definition and the univalence of the local leaf $L(\hat{c}, D_\delta(a))$. For $k \leq j$ one has $w_{-k} = y_{-k}$. The latters are not critical points of f, since $\hat{y} \in \Pi_a$. Hence, $\hat{w} \in \Pi_a$.

By (1.14),

$$\beta(\hat{a}, \hat{w}) = \beta(\hat{a}, \hat{y}) + \beta(\hat{y}, \hat{w}). \tag{2.8}$$

We show that

$$\beta(\hat{y}, \hat{w}) = \beta_{\hat{a},\hat{c}}(y_{-j}). \tag{2.9}$$

This together with (2.8) implies (2.7). One has

$$\beta(\hat{y}, \hat{w}) = \beta(\hat{f}^{-j}(\hat{y}), \hat{f}^{-j}(\hat{w})) = \beta(\hat{t}, \hat{\zeta}), \ \hat{t} = \hat{f}^{-j}(\hat{y}),$$

by (1.15) and since $\pi_0(\hat{t}) = \pi_0(\hat{\zeta}) = y_{-j}$. The point $\hat{t}$ lies in the local leaf $L(\hat{a}, D_\delta(a))$. This follows from the choice of j ($t_{-s} = y_{-j-s} \in D_\delta(a)$ for any $s \geq 0$) and the invariance of $D_\delta(a)$ under the inverse branch f^{-1} fixing a (the condition of the proposition). Thus,

$$\hat{t} \in L(\hat{a}, D_\delta(a)),\ \hat{\zeta} \in L(\hat{c}, D_\delta(a)),\ \pi_0(\hat{t}) = \pi_0(\hat{\zeta}) = y_{-j}. \text{ Therefore,}$$

$$\beta(\hat{y}, \hat{w}) = \beta(\hat{t}, \hat{\zeta}) = \beta_{\hat{a},\hat{c}}(y_{-j}),$$

by (1.18). This proves (2.9) and (2.7). □

Corollary 2.4 *Let f, a, Π_a be as in Proposition 2.3. The closure*

$$\mathcal{B} = \overline{\{\beta(\hat{a}, \hat{y}) \mid \hat{y} \in \Pi_a\}} \tag{2.10}$$

is an additive semigroup in $\mathbb{R}$.

Proof We have to show that for any given $\hat{y}, \hat{c} \in \Pi_a$ and $\varepsilon > 0$ there exists a $\hat{w} \in \Pi_a$ such that $|\beta(\hat{a}, \hat{w}) - (\beta(\hat{a}, \hat{y}) + \beta(\hat{a}, \hat{c}))| \leq \varepsilon$. Let y_{-j} and $\hat{w}$ be as in the above proposition. Then $\hat{w}$ is a one we are looking for, whenever j is large enough. Indeed, the previous difference equals $\beta_{\hat{a},\hat{c}}(y_{-j}) - \beta_{\hat{a},\hat{c}}(a)$, by (2.7) and since $\beta(\hat{a}, \hat{c}) = \beta_{\hat{a},\hat{c}}(a)$ by definition. The latter difference tends to 0, as $j \to +\infty$, since $y_{-j} \to a$. This proves the corollary. □

We use the following elementary property of additive semigroups.

Proposition 2.5 *Let $\mathcal{B} \subset \mathbb{R}$ be an additive semigroup such that for any $\varepsilon > 0$ it contains a pair of at most ε- close distinct elements. Then for any $M \in \mathbb{R} \backslash 0$ the semigroup $\mathcal{B}_M = \mathcal{B} + \mathbb{Z}M$ is dense in $\mathbb{R}$.*

Proof Fix a $\varepsilon > 0$ and a pair $A, B \in \mathcal{B}$, $A \neq B$, $\varepsilon' = |A - B| \leq \varepsilon$. The elements A and B generate a semigroup that contains arithmetic progressions with step ε' and arbitrarily large lengths: for any $m \in \mathbb{N}$ the set $\{sA + (m-s)B \mid 0 \leq s \leq m\}$ is such a progression. Fix a m such that the ends mA, mB of the latter progression bound an interval of length at least M. The images of the progression by translations by nM, $n \in \mathbb{Z}$, form a ε'- net on $\mathbb{R}$. Therefore, the set $\mathcal{B}_M$ from the proposition contains a ε'- net on $\mathbb{R}$ with arbitrarily small ε'. Hence, it is dense. The proposition is proved. □

By definition, one has

$$\mathcal{B} + \mathbb{Z} \ln |f'(a)| \subset \overline{\mathcal{B}f}. \tag{2.11}$$

We show that the semigroup $\mathcal{B}$ contains distinct elements arbitrarily close to each other. Then applying Proposition 2.5 to $M = \ln |f'(a)|$ together with (2.11) implies Lemma 2.2.

As it is shown below, the above statement on $\mathcal{B}$ is implied by (2.7) and the following

Lemma 2.6 (Main Technical Lemma) *Let f be a rational function that does not belong to the list (1.1). Let $a \in \overline{\mathbb{C}}$ be some its repelling fixed point that is not branch-exceptional, $\hat{a} \in \mathcal{A}_f$ be its fixed orbit, Π_a be the set from (2.3). There exists a pair of points $\hat{y}, \hat{c} \in \Pi_a$ such that for any $N \in \mathbb{N}$*

$$\beta_{\hat{a},\hat{c}}|_{\{y_{-j} \ \mid \ j \geq N\}} \not\equiv const. \tag{2.12}$$

The proof of Lemma 2.6 (given in the rest of the section) uses essentially the analyticity of basic cocycle.

Proof of Lemma 2.2 modulo Lemma 2.6. It suffices to show that the semigroup $\mathcal{B}$ contains pairs of arbitrarily close distinct elements (see the above discussion). Let $\hat{y}, \hat{c} \in \Pi_a$ be as in Lemma 2.6. The value $B' = \beta(\hat{a}, \hat{y}) + \beta(\hat{a}, \hat{c})$ is contained in $\mathcal{B}$ (Corollary 2.4). On the other hand, for any j and $\hat{w}$ as in (2.7) the value $\beta(\hat{a}, \hat{w})$ also belongs to $\mathcal{B}$. It differs from B' by $\beta_{\hat{a},\hat{c}}(y_{-j}) - \beta_{\hat{a},\hat{c}}(a)$, which tends to 0, as $j \to +\infty$ (see the proof of Corollary 2.4). The latter difference is nonzero for an infinite number of values of j (Lemma 2.6). Thus, the previously constructed elements B' and $\beta(\hat{a}, \hat{w})$ of the semigroup $\mathcal{B}$ can be made distinct and arbitrarily close to each other. This proves Lemma 2.2. □

Proof of Theorem 1.49 modulo Lemma 2.6. To prove the density of the set $\mathcal{S}$, see (2.1), with $S = S_{\hat{a},0}$, we use Lemma 2.2 and Proposition 1.41. For any $h \in \mathcal{B}f$ we construct a sequence $\{\hat{a}^m\}|_{m\in\mathbb{N}} \subset \Pi_a$ such that $(\hat{a}^m, h) \in \mathcal{S}$ for all m and $(\hat{a}^m, h) \to (\hat{a}, h)$. This together with the density of $\mathcal{B}f$ (Lemma 2.2) implies that $\mathcal{S}$ accumulates to all the points $(\hat{a}, r)$, $r \in \mathbb{R}$, and hence, to the horospheres through these points (Proposition 1.41). The latter horospheres saturate the whole leaf $H(\hat{a})$, thus, $H(\hat{a}) \subset \overline{\mathcal{S}}$. This together with the discussion from the beginning of the section implies Theorem 1.49.

Fix a $h \in \mathcal{B}f$. By definition, there exist a $\hat{y} \in \Pi_a$ and a $l \in \mathbb{Z}$ such that $h + l \ln|f'(a)| = \beta(\hat{a}, \hat{y})$, i.e., $(\hat{y}, h + l\ln|f'(a)|) \in S_{\hat{a},0}$, see (2.6). Then for any $m \in \mathbb{N}$

$$q_m = (\hat{y}, h + (l-m)\ln|f'(a)|) \in S_{\hat{a},-m\ln|f'(a)|} \subset \mathcal{S},$$

by (2.2), where m is replaced by 0 and h is replaced by $-m\ln|f'(a)|$. For any $m \geq l$ set

$$\hat{a}^m = \hat{f}^{m-l}(\hat{y}), \alpha_m = (\hat{a}^m, h) = \hat{f}^{m-l}(q_m).$$

The sequence $\{\hat{a}^m\}_{m\geq l}$ is a one we are looking for. Indeed, $\alpha_m \in \mathcal{S}$, since $\mathcal{S}$ is $\hat{f}$- invariant and $q_m \in \mathcal{S}$. One has $a_0^m = f^{m-l}(y_0) = f^{m-l}(a) = a$, whenever $m \geq l$. Set $\hat{b}^m = \hat{y}$. Let V be a neighborhood of a such that the local leaf $L(\hat{y}, V)$ is univalent over V. The sequence $\hat{a}^m$ converges to $\hat{a}$ in $\mathcal{A}_f$, by Corollary 1.36 applied to these $\hat{b}^m$ and V. This together with Proposition 1.42 proves the convergence $\alpha_m \to (\hat{a}, h)$ and finishes the proof of Theorem 1.49. □

2.2 Proof of the Main Lemma 2.6 modulo technical details

Let f be a rational function. Let a be some its repelling fixed point that is not branch-exceptional, $\hat{a} \in \mathcal{A}_f$ be its fixed orbit, Π_a be the set from (2.3). Fix a small neighborhood U of a where f is univalent and such that $f(U) \supset U$. The branch of f^{-1} fixing a is holomorphic on U, sends U to itself and its iterations converge to a uniformly on compact subsets of U. (In particular, the local leaf $L(\hat{a}, U)$ is univalent over U.) Therefore, the linearizing coordinate of f at a extends up to a conformal coordinate on U. In addition, we will assume for a technical reason that U is convex (e.g., a disk) in the linearizing coordinate.

Definition 2.7 Let f, $\hat{a}$, U be as above. Let $\hat{z} = (z_0, z_{-1}, \dots) \in L(\hat{a})$, $N > 0$ be such that $z_{-j} \in U$ for any $j \geq N$. (The number N is not necessarily the minimal one with this property.) The backward orbit $z_{-N}, z_{-N-1}, \dots$ is called a *tail of* $\hat{z}$. If N is the minimal number as above, then the tail is called *complete*. (Equivalently, a (complete) tail is a (maximal) backward orbit representing a point of $L(\hat{a}, U)$.)

Lemma 2.8 *Let f be a rational function that does not belong to the list (1.1), a, Π_a be as above. There exists a point $\hat{c} \in \Pi_a$ such that*

$$\beta_{\hat{a},\hat{c}} \not\equiv const \textit{ in a neighborhood of } a. \tag{2.13}$$

This lemma is proved in the next subsection. Fix a $\hat{c} \in \Pi_a$ satisfying (2.13). We prove the statement of Lemma 2.6 for this $\hat{c}$: there exists a $\hat{y} \in \Pi_a$ such that $\beta_{\hat{a},\hat{c}} \not\equiv const$ along any tail of $\hat{y}$. Without loss of generality we consider that the local leaf $L(\hat{c}, U)$ is univalent over U. One can achieve this by shrinking U. Recall that the function $\beta_{\hat{a},\hat{c}}$ is real-analytic on U. We consider the auxiliary analytic subset $A \subset U$:

$A =$ the intersection of all the analytic subsets in U that contain some tail of each $\hat{y} \in \Pi_a$.

Proposition 2.9 *Let f, a, U, be as at the beginning of the subsection, A be as above. The set A is either the whole U, or a line interval through a in the linearizing chart of f on U.*

Remark 2.10 If A is a line interval, then $f'(a) \in \mathbb{R}$. This follows from definition and the f-invariance of the germ of A at a (see Claim 2 in 2.4 for a stronger invariance statement).

Recall that the linearizing coordinate on U lifts to the local leaf $L(\hat{a}, U)$ and extends up to a global affine coordinate on $L(\hat{a})$ (Example 1.24). We consider the auxiliary conformal mapping

$$\psi_{\hat{c}} : U \to L(\hat{c}, U) \subset L(\hat{a}) = \mathbb{C}, \ \psi_{\hat{c}} = (\pi_0|_{L(\hat{c},U)})^{-1} : \text{ one has } \beta_{\hat{a},\hat{c}} \equiv -\ln|\psi'_{\hat{c}}|, \tag{2.14}$$

the derivative is taken in the above coordinates. This follows from (1.17).

Proposition 2.11 *In the conditions of Proposition 2.9 let A be a line interval. Let $\hat{A} \subset L(\hat{a})$ denote the line whose intersection with the local leaf $L(\hat{a}, U)$ is projected to A. For any $\hat{c} \in \Pi_a$ such that the local leaf $L(\hat{c}, U)$ is univalent over U the mapping $\psi_{\hat{c}}$ from (2.14) sends A to $\hat{A}$.*

Proositions 2.9 and 2.11 will be proved in 2.4.

Proof of Lemma 2.6 modulo Propositions 2.9 and 2.11. Let f do not belong to the list (1.1). Let a, $\hat{a}$, U, A, $\hat{c}$ be as above. We prove Lemma 2.6 by contradiction. Suppose the contrary: $\beta_{\hat{a},\hat{c}} \equiv const$ along some tail of each $\hat{y} \in \Pi_a$. Then $\beta_{\hat{a},\hat{c}}|_A \equiv \beta_{\hat{a},\hat{c}}(a) = \beta(\hat{a}, \hat{c})$, by definition, analyticity and since all the tails have the common limit a. If $A = U$, then the latter identity contradicts (2.13). Thus, A is a line interval (Proposition 2.9) and $\beta_{\hat{a},\hat{c}}|_A \equiv const$. Therefore, the conformal mapping $\psi_{\hat{c}}$ from (2.14) sends the line A to the line $\hat{A}$ (Proposition 2.11), and its derivative in the linearizing chart has constant module along A. The next proposition shows that $\psi_{\hat{c}}$ is an affine mapping.

Proposition 2.12 *Let ψ be a conformal mapping of one domain of $\mathbb{C}$ onto another one. Let ψ map a line interval A to a line and the module of its derivative be constant along A. Then ψ is an affine mapping.*

Proof There exists a complex affine mapping that coincides with ψ on A (the constance of the module of the derivative on A and the linearity of A and its image). Then it coincides with ψ everywhere (the uniqueness of analytic extension). This proves the proposition. $\Box$

Thus, the mapping $\psi_{\hat{c}}$ is affine. Therefore, its derivative is constant on U, and hence, $\beta_{\hat{a},\hat{c}} \equiv const$ on U, by (2.14), - a contradiction to (2.13). Lemma 2.6 is proved. $\Box$

Remark 2.13 The above arguments show that if $\beta_{\hat{a},\hat{c}}|_A \equiv const$, then $\beta_{\hat{a},\hat{c}} \equiv const$. Thus, if the former holds true for all $\hat{c} \in \Pi_a$, then the function f belongs to the list (1.1) (Lemma 2.8). Generically, $A = U$. There exist functions f that do not belong to the list (1.1) but for which A is a line interval, see the next example.

Example 2.14 Consider a quadratic polynomial $f(x) = cx(1-x)$, $c > 4$. It has a repelling fixed point $a = 0$. It is well-known that there exists a Cantor set in $[0,1]$ containing 0 and completely invariant (i.e., coinciding with its preimage) under the complex dynamics of f. The corresponding set A is an interval of the real line, since each complex preimage of 0 under arbitrary iterate of f is real: it is contained in the above Cantor set. On the other hand, the polynomial f does not belong to the list (1.1).

2.3 Nonconstance of basic cocycles. Proof of Lemma 2.8

Fix an affine Euclidean metric g on $L(\hat{a})$. We prove the lemma by contradiction. Suppose the contrary: $\beta_{\hat{a},\hat{c}} \equiv const$ in a neighborhood of a for all $\hat{c} \in \Pi_a$. We show (the next proposition) that the latter constant is zero. Afterwards we show (Proposition 2.16) that the metric g is projected to some well-defined (singular) metric on $\overline{\mathbb{C}}$ and extends up to a continuous family of affine Euclidean metrics on the leaves of $\mathcal{A}'_f$. Hence, f belongs to the list (1.1), by Theorem 1.47, - a contradiction to the conditions of Lemma 2.8.

Proposition 2.15 *Let f, a, $\hat{a}$, Π_a be as at the beginning of the previous subsection. Let $\hat{c} \in \Pi_a$ be such that $\beta_{\hat{a},\hat{c}} \equiv const$ in a neighborhood of a. Then $\beta_{\hat{a},\hat{c}} \equiv 0$.*

Proof Let U be a neighborhood of a such that the local leaves $L(\hat{a}, U)$ and $L(\hat{c}, U)$ are univalent over U. Consider the mapping

$$\psi : L(\hat{a}, U) \to L(\hat{c}, U), \ \psi = (\pi_0|_{L(\hat{c},U)})^{-1} \circ \pi_0.$$

The module of the derivative of ψ in the Euclidean metric g on $L(\hat{a})$ is constant, since its logarithm equals $-\beta_{\hat{a},\hat{c}} \equiv const$ by (1.17). Therefore, ψ extends up to an affine automorphism $\psi : L(\hat{a}) \to L(\hat{a})$. If the module of its derivative equals 1, then $\beta_{\hat{a},\hat{c}} \equiv 0$, and the statement of the proposition follows immediately. Otherwise ψ has a fixed point (denote it q) that is either an attractor, or a repeller. The function $\pi_0 : L(\hat{a}) \to \overline{\mathbb{C}}$ is ψ- invariant by construction and analyticity. Therefore, it is constant in a neighborhood of q (analyticity). Hence, $\pi_0|_{L(\hat{a})} \equiv const$, - a contradiction. The proposition is proved. $\Box$

Proposition 2.16 *Let f, a, $\hat{a}$, Π_a be as at the beginning of the previous subsection. Let*

$$\beta_{\hat{a},\hat{c}} \equiv 0 \textit{ for any } \hat{c} \in \Pi_a. \tag{2.15}$$

Consider arbitrary triple (W, W_1, W_2), where $W \subset \overline{\mathbb{C}}$ is a simply connected domain intersecting the Julia set of f, $W_1, W_2 \subset L(\hat{a})$ are local leaves 1-to-1 projected to W by π_0. Let g be an affine Euclidean metric on $L(\hat{a})$. Then the pushforwards $(\pi_0)_*(g|_{W_i})$ of the metrics $g|_{W_i}$, $i = 1, 2$, to W coincide.

In the proof of the proposition we use the following

Proposition 2.17 *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational function, $a \in \overline{\mathbb{C}}$ be some its repelling fixed point that is not branch-exceptional. Let $W \subset \overline{\mathbb{C}}$ be an arbitrary simply connected domain intersecting the Julia set. There exist a $n \in \mathbb{N}$ and a $z \in W$ such that $f^n(z) = a$ and $(f^n)'(z) \neq 0$.*

Proof Fix a nonperiodic backward orbit $\hat{q} = (q_0 = a, q_{-1}, \dots)$ avoiding critical points. Then q_{-j} is not a postcritical point, whenever j is large enough. Indeed, otherwise the nonperiodic (and hence, *infinite*) *backward* orbit $\hat{q}$ is contained in the finite union of *forward* postcritical orbits, which is impossible. Fix three distinct points q_{-j_1}, q_{-j_2}, q_{-j_3}, $j_1 < j_2 < j_3$. The union $\mathcal{U} = \cup_{m \in \mathbb{N} \cup 0} f^m(W)$ contains some of them. Otherwise, the family $f^m : W \to \overline{\mathbb{C}}$ of meromorphic functions avoids three distinct values and is normal by Montel's theorem ([12], p.52). Hence, $W \cap J = \emptyset$, - a contradiction. This implies that there exist a $m \in \mathbb{N}$ and a $z \in W$ such that $f^m(z)$ coincides with some of the three above values. Hence, $f^n(z) = a$ for $n = m + j_3$. By construction, $(f^n)'(z) \neq 0$, since no q_{-j} is a critical point. This proves the proposition. □

Proof of Proposition 2.16. Let $z \in W$ and $n \in \mathbb{N}$ be as in Proposition 2.17,

$$\hat{z}^i = (\pi_0|_{W_i})^{-1}(z) \in W_i, \ \hat{y}^i = \hat{f}^n(\hat{z}^i), \ i = 1, 2.$$

Then $\hat{y}^i \in \Pi_a$. Indeed, $\hat{y}^i \in L(\hat{a})$, since $\hat{z}^i = \hat{f}^{-n}(\hat{y}^i) \in W_i \subset L(\hat{a})$. One has $\pi_0(\hat{y}^i) = f^n(z) = a$. The germ of the projection $\pi_0 : (L(\hat{a}), \hat{y}^i) \to (\overline{\mathbb{C}}, a)$ has nonzero derivative at $\hat{y}^i$. Indeed, this germ is equal to the composition $f^n \circ \pi_0 \circ \hat{f}^{-n}$ of the following germs:

$$\hat{f}^{-n} : (L(\hat{a}), \hat{y}^i) \to (L(\hat{a}), \hat{z}^i), \quad \pi_0 : (L(\hat{a}), \hat{z}^i) \to (\overline{\mathbb{C}}, z), \quad f^n : (\overline{\mathbb{C}}, z) \to (\overline{\mathbb{C}}, a).$$

Each of them has nonzero derivative at the corresponding base point, since $\hat{f}^{-n}$ is affine, each $\hat{z}^i$ is contained in a univalent leaf W_i and $(f^n)'(z) \neq 0$. Thus, the derivative of the composition at $\hat{y}^i$ is also nonzero. The statement of Proposition 2.16 is equivalent to the identity $\beta_{\hat{z}^1, \hat{z}^2} \equiv 0$, by the definition of basic cocycle. One has $\beta_{\hat{z}^1, \hat{z}^2} = \beta_{\hat{y}^1, \hat{y}^2} = \beta_{\hat{a}, \hat{y}^2} - \beta_{\hat{a}, \hat{y}^1} \equiv 0$, by the invariance of basic cocycle, the cocycle identity and (2.15). This proves Proposition 2.16. □

Proof of Lemma 2.8. Recall that we assume that $\beta_{\hat{a}, \hat{c}} \equiv const$, hence, $\beta_{\hat{a}, \hat{c}} \equiv 0$ for all $\hat{c} \in \Pi_a$ (Proposition 2.15). Let $\hat{z}, \hat{w} \in L(\hat{a})$ be arbitrary two points with $z_0 = w_0$ such that $(\pi_0|_{L(\hat{a})})'(\hat{z}), (\pi_0|_{L(\hat{a})})'(\hat{w}) \neq 0$. Consider the restrictions to $T_{\hat{z}} L(\hat{a})$ and $T_{\hat{w}} L(\hat{a})$ of the affine metric g of the leaf $L(\hat{a})$. The latter are projected by π_0 to the same metric on $T_{z_0} \overline{\mathbb{C}}$. This follows by choosing a small neighborhood W of z_0 such that the local leaves $L(\hat{z}, W)$ and $L(\hat{w}, W)$ are univalent and applying Proposition 2.16. Those pairs of points $\hat{z}$, $\hat{w}$ form an open and dense subset in the space of pairs of points in $L(\hat{a})$ with coinciding projections. Hence, the metric g of the whole leaf $L(\hat{a})$ is projected by π_0 to a well-defined (maybe singular) conformal metric on $\overline{\mathbb{C}}$. The only possible singularities of the projected metric are the so-called *persistently-critical values*: those $z \in \overline{\mathbb{C}}$, whose preimages under the projection $\pi_0|_{L(\hat{a})}$

either are empty or contain only critical points of the projection. (There can be at most 4 persistently-critical values, by Ahlfors' five island theorem, see [18], theorem VI.8.) Lifting the projected metric to other leaves extends g up to a continuous family of conformal metrics on all the leaves of $\mathcal{A}'_f$ (that a priori may have isolated singularities on some leaves). The lifted metrics are affine on all the leaves, as is g, by the density of the leaf $L(\hat{a})$ (Proposition 1.38) and the continuity of the affine structure family. Therefore, f belongs to the list (1.1), - a contradiction. This proves Lemma 2.8. □

2.4 The common level of basic cocycles. Proof of Propositions 2.9 and 2.11

Everywhere below we suppose that $A \neq U$. This implies that there exists a real-analytic curve containing a tail of each $\hat{y} \in \Pi_a$. First we show that A is a finite union of line intervals (Claim 1). Afterwards we prove Propositions 2.9 and 2.11, using the invariance of A under appropriate inverse branches (Claims 2 and 3).

Claim 1. *The set A is a finite union of line intervals intersecting at a with ends in ∂U. It contains a complete tail of each $\hat{y} \in \Pi_a$.*

Proof Any tail is an orbit under the iterations of the multiplication by $(f'(a))^{-1}$ in the linearizing chart. Therefore, $\arg f'(a) = \pi \frac{p}{q} \in \mathbb{Q}$: otherwise no tail can be contained in a real-analytic curve, and $A = U$, - a contradiction. Therefore, for any $\hat{y} \in \Pi_a$ there exists a union $\Lambda_{\hat{y}}$ of q lines through a such that each tail of $\hat{y}$ is contained there and intersects each line at a sequence of points converging to a. The intersection $\Lambda_{\hat{y}} \cap U$ is a union of q intervals through a with ends in ∂U, since U is convex by assumption. For any $\hat{y} \in \Pi_a$ the latter intersection lies in A and contains the complete tail of $\hat{y}$, by the two previous statements and analyticity. Thus, A is a union of intervals through a. This union is finite by analyticity. Claim 1 is proved. □

Claim 2. *The set A is invariant under the branch $f^{-1} : U \to U$ fixing a.*

Proof The f^{-1}- invariance of A follows from definition and the invariance of U. □

Claim 3. *Let $n \in \mathbb{N}$, f^{-n} be an arbitrary germ of holomorphic inverse branch at a such that $f^{-n}(a) \in U$. Then f^{-n} sends the germ of A at a to A.*

Proof Fix a neighborhood $V \subset U$ of a such that f^{-n} is holomorphic on V and $f^{-n}(V) \subset U$. This defines the holomorphic branches

$$f^{-j} = f^{n-j} \circ f^{-n} : V \to \overline{\mathbb{C}} \text{ for all } j \in \mathbb{N} : \tag{2.16}$$

for $j \leq n$ the function f^{n-j} is well-defined on $\overline{\mathbb{C}}$; for $j > n$ we take $f^{n-j} : U \to U$ to be the holomorphic branch fixing a. Then $f^{-j}(V) \subset U$ for all $j \geq n$ and $f^{-j} \to a$ uniformly on compact subsets in V, as $j \to +\infty$. Therefore, $\hat{q} = (a, f^{-1}(a), f^{-2}(a) \dots) \in \Pi_a$ and $f^{-n}(a)$ is contained in the complete tail of $\hat{q}$. Thus, $f^{-n}(a) \in A$ (Claim 1).

Fix an arbitrary $\hat{y} \in \Pi_a$. For any l large enough $y_{-l} \in V$. For these l the values $f^{-j}(y_{-l})$ with $j \in \mathbb{N}$ are defined by (2.16). Those of them with $j \geq n$ form a tail of the backward orbit

$$(y_0 = a, y_{-1}, \dots, y_{-l}, f^{-1}(y_{-l}), f^{-2}(y_{-l}), \dots) \in \Pi_a.$$

Therefore, $f^{-n}(y_{-l}) \in A$ (Claim 1). This holds true for arbitrary $\hat{y} \in \Pi_a$ and any l large enough (dependently on $\hat{y}$). Thus, f^{-n} maps some tail of arbitrary $\hat{y} \in \Pi_a$ to A. Hence, f^{-n} sends the germ of A at a to A, by definition and analyticity. This proves Claim 3. □

Proof of Proposition 2.9. Suppose the contrary: $A \neq U$ and A is not a single interval. Hence, it is a finite union of line intervals through a (Claim 1) and contains at least two distinct line intervals I_1, I_2 intersecting transversely at a. Fix an arbitrary $\hat{y} \in \Pi_a$. Let $(y_{-n}, y_{-n-1}, \dots)$ be some its tail in U, f^{-n} be the corresponding inverse branch germ at a: $f^{-n}(a) = y_{-n}$. Then f^{-n} sends the germ of A at a to A (Claim 3). It transforms the germs of the intervals $I_s \subset A$, $s = 1, 2$, into germs of two analytic curves intersected transversely at $y_{-n} \neq a$. Thus, A contains the latter germs and hence, cannot be a finite union of line intervals through a, - a contradiction to Claim 1. Proposition 2.9 is proved. □

Proof of Proposition 2.11. The line $\hat{A}$ is $\hat{f}$- invariant (Claim 2). Let $(c_{-n}, c_{-n-1}, \dots)$ be a tail of $\hat{c}$, thus, $\hat{f}^{-n}(\hat{c}) \in L(\hat{a}, U)$. By definition, $\psi_{\hat{c}}(a) = \hat{c}$. Let f^{-n} be the inverse branch germ at a sending a to $c_{-n} \in U$. The germ at a of the mapping $\hat{f}^{-n} \circ \psi_{\hat{c}}$ is the composition of two mappings: the above germ f^{-n} and $\pi_0^{-1} : U \to L(\hat{a}, U)$. The former sends the germ of A at a to A (Claim 3). The latter sends A to $\hat{A}$ by definition. Therefore, $\hat{f}^{-n} \circ \psi_{\hat{c}}(A) \subset \hat{A}$, and hence, $\psi_{\hat{c}}(A) \subset \hat{A}$. This proves Proposition 2.11. □

3 Accumulation to the horospheres over repellers. Proof of Theorem 1.53

3.1 The plan of the proof of Theorem 1.53

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a critically-nonrecurrent rational mapping, $L \subset \mathcal{A}_f$ be a leaf of the corresponding affine lamination whose projection $p(L) \subset \mathcal{A}_f^n$ does not lie in a leaf associated to a parabolic periodic point. Let $H \subset \mathcal{H}_f$ be the corresponding hyperbolic leaf. We show that there exists a repelling periodic point $a \in \overline{\mathbb{C}}$ of f that is not branch-exceptional and such that for each horosphere $S \subset H$ the union of its images under all the forward and backward iterations of $\hat{f}$ accumulates to some point in $H(\hat{a})$, and hence, to the horosphere passing through this point. This will prove Theorem 1.53.

As in the proof of Theorem 1.49, it suffices to prove the above statement for just one horosphere $S \subset H$ (let us fix it). First we construct appropriate sequence $n_k \in \mathbb{N}$, $n_k \to \infty$, as $k \to \infty$, a disk $V \subset \overline{\mathbb{C}}$ intersecting the Julia set (we fix a Hermitian Euclidean metric on V), appropriate local leaves $L_k \subset \hat{f}^{-n_k}(L)$ over V such that

$$\text{for any } k \in \mathbb{N} \text{ the local leaf } L_k \text{ is univalent over } V. \tag{3.1}$$

For any $w \in V$ we set

$$\hat{w}^k = (\pi_0|_{L_k})^{-1}(w), S^k = \hat{f}^{-n_k}(S), \ h_k(w) = \text{ the height of } S^k \text{ over } \hat{w}^k. \tag{3.2}$$

We show that

$$h_k \to -\infty \text{ uniformly on compact sets in } V, \text{ as } k \to +\infty. \tag{3.3}$$

Proposition 3.1 *Let V, L_k, S^k be as above and satisfy (3.1) and (3.3). Let $a \in V$ be an arbitrary repelling periodic point, $\hat{a}$ be its periodic orbit, $\hat{a}^k$ be as in (3.2), $\alpha^k = (\hat{a}^k, \theta_k) \in S^k$ be the point of the horosphere S^k over $\hat{a}^k$. There exist a $h \in \mathbb{R}$ and a sequence $l_k \in \mathbb{N}$, $l_k \to \infty$, as $k \to \infty$, such that, after passing to a subsequence, one has*

$$\hat{f}^{l_k}(\alpha^k) \to (\hat{a}, h) \in H(\hat{a}), \ \text{as } k \to \infty. \tag{3.4}$$

Proof Let $s \in \mathbb{N}$ be the period of a, $\lambda = |(f^s)'(a)| > 1$. Then for any $m \in \mathbb{N}$ one has

$$\hat{f}^{ms}(\alpha^k) = (\hat{f}^{ms}(\hat{a}^k), h_{m,k}),\ h_{m,k} = \theta_k + m \ln \lambda. \text{ Set } m_k = -[\frac{\theta_k}{\ln \lambda}].$$

By construction, the sequence $h_{m_k,k}$ is bounded: $0 \leq h_{m_k,k} \leq \ln \lambda$. Passing to a subsequence one can achieve that it converges to some $h \in \mathbb{R}$. Set $l_k = m_k s$. One has $\theta_k \to -\infty$, by (3.3); thus, $m_k, l_k \to +\infty$. Hence, $\hat{f}^{l_k}(\hat{a}^k) \to \hat{a}$, by (3.1) and Corollary 1.36. Then $\hat{f}^{l_k}(\alpha^k) = (\hat{f}^{l_k}(\hat{a}^k), h_{m_k,k}) \to \alpha = (\hat{a}, h) \in H(\hat{a})$, by Proposition 1.42. This proves (3.4). □

Proof of Theorem 1.53 modulo (3.1) and (3.3). Fix an arbitrary repelling periodic point $a \in V$ (these points are dense in the Julia set). It is not branch-exceptional: there are infinitely many univalent local leaves L_k over V, by (3.1) and (3.3). Let l_k, α^k, h be as in Proposition 3.1. Then the horospheres $\hat{f}^{l_k}(S^k)$ accumulate to the horosphere through $(\hat{a}, h)$ (Proposition 3.1). This proves Theorem 1.53. □

In the proof of (3.1) and (3.3) we use the following well-known theorem.

Theorem 3.2 *(Mañe, [15], theorem II). Let f be a rational function, $z \in \overline{\mathbb{C}}$ be a point that is neither an attractive, nor a parabolic periodic point and does not belong to the ω- limit set of a recurrent critical point. Then for any $\varepsilon > 0$ there exists a neighborhood $V = V(z)$ such that for any $m \in \mathbb{N}$ each connected component of the preimage $f^{-m}(V)$ has diameter less than ε.*

First we construct V, n_k and L_k in the case, when $L \subset \mathcal{A}_f^l$ (the next subsection). The construction in the opposite case, which is given in 3.3, is similar but slightly more technical.

3.2 Case when $L \subset \mathcal{A}_f^l$

Fix a $\hat{x} \in L$, $\pi_0(\hat{x}) \in J = J(f)$. The above-mentioned construction is based on the following

Lemma 3.3 *Let f be a critically-nonrecurrent rational function, $L \subset \mathcal{A}_f^l$ be a leaf that is not associated to a parabolic periodic point. Let $\hat{x}$ be as above. There exist a sequence $n_k \to +\infty$, a point $b \in J(f)$ (that is not a parabolic periodic point) and a neighborhood $V = V(b) \subset \overline{\mathbb{C}}$ such that $x_{-n_k} \to b$ and for any $k \in \mathbb{N}$ the local leaf $L_k = L(\hat{f}^{-n_k}(\hat{x}), V)$ is univalent over V.*

Proof Recall that f is critically-nonrecurrent. This implies that there is a natural ordering of some pairs of critical points of f that lie in the Julia set of f. Namely, given two critical points $c_1 \neq c_2 \in J$, we say that $c_1 > c_2$, if either c_2 is the image of c_1 under some positive iteration of f, or c_2 belongs to the ω- limit set of c_1. This ordering has the transitivity property: if $c_1 > c_2$ and $c_2 > c_3$, then $c_1 > c_3$ (critical nonrecurrence and absence of periodic critical points in the Julia set).

Let us consider two different cases.

Case 1): the limit set of $\{x_{-n}\}|_{n \in \mathbb{N}}$ contains no critical point of f. Choose a subsequence $n_1 < n_2 < \dots$ so that x_{-n_k} converge to some point $b \in \overline{\mathbb{C}}$ that is not a parabolic periodic point. It is possible, since the Riemann sphere is compact and the leaf $L = L(\hat{x})$ is not a leaf associated to a parabolic periodic point. Then for any $\varepsilon > 0$ there exists a neighborhood $V = V(b)$ such that for any $m \in \mathbb{N} \cup 0$ each connected component of the preimage $f^{-m}(V)$ has diameter less than ε (by Mañe's Theorem). Fix these ε and V so that

$$\varepsilon < dist(x_{-n}, c) \text{ for any critical point } c \text{ of } f \text{ and any } n \in \mathbb{N} \text{ large enough.} \quad (3.5)$$

One can achieve this by assumption: x_{-n} accumulate to no critical point. Without loss of generality we assume that $x_{-n_k} \in V$ for any k and inequality (3.5) holds true for all $n \geq n_1$. One can achieve this by removing all "too small" n_k by the convergence $x_{-n_k} \to b \in V$. For any $m \in \mathbb{N}$ set

$$V_{-m,k} = \text{ the connected component containing } x_{-m-n_k} \text{ of } f^{-m}(V). \qquad (3.6)$$

Each $V_{-m,k}$ contains no critical point by (3.5) and since $diam V_{-m,k} < \varepsilon$. This implies that each local leaf $L_k = L(\hat{f}^{-n_k}(\hat{x}), V)$ is univalent over V.

Case 2): the sequence x_{-n} accumulates to some critical point of f. The limit critical points lie in the Julia set, as does x_0. Let b be a maximal limit critical point. Fix a $\varepsilon > 0$ satisfying (3.5) for any critical point $c > b$ and a neighborhood $V = V(b)$ satisfying the statement of Mañe's Theorem. Without loss of generality we consider that if V contains an image $f^n(c')$, $n \in \mathbb{N}$, of a critical point c', then $c' > b$ (hence, c' is not a limit critical point). One can achieve this by shrinking V, by the definition of the order on critical points. Fix an arbitrary subsequence n_k so that $x_{-n_k} \to b$, as $k \to \infty$, and $x_{-n_k} \in V$ for all k. Then each above local leaf L_k is univalent over V. Indeed, for any $m \in \mathbb{N}$ the domain $V_{-m,k}$ from (3.6) contains no critical point $c > b$ (by the choice of ε and Mañe's Theorem). It contains no other critical point by the choice of V. Lemma 3.3 is proved. □

Let n_k, b, V and L_k be as in the above lemma. Statement (3.1) follows from the lemma. In the proof of (3.3) we use the following

Proposition 3.4 *For any rational function f, any simply connected domain $V \subset \overline{\mathbb{C}}$ (we fix a Hermitian Euclidean metric on it), any family of univalent local leaves $L_k \subset \mathcal{A}_f$ over V, any family of horospheres S^k in the corresponding hyperbolic leaves in $\mathcal{H}_f$, the height functions $h_k(w)$, see (3.2), are equicontinuous on compact sets.*

Proof Without loss of generality we consider that V is unit disk and $b = 0$. Each local leaf L_k is affine isomorphic to a domain $V^k \subset \mathbb{C}$. Indeed, the ambient leaf is isomorphic to the quotient of $\mathbb{C}$ by a discrete group of Euclidean isometries. The local leaf L_k is simply connected, as is V, and contains no singularities (Definition 1.35). By definition, one has

$$h_k = -\ln|\tau_k'| + const(k), \text{ where } \tau_k = (\pi_0|_{L_k})^{-1} : V \to L_k = V^k \subset \mathbb{C}.$$

The functions τ_k are univalent and can be normalized so that $\tau_k(0) = 0$, $\tau_k'(0) = 1$, by applying affine transformations in the image. The latter logarithms remain unchanged (up to additive constants). Thus normalized univalent functions form a normal family. This implies the equicontinuity of the heights h_k on compact sets and proves the proposition. □

Proof of (3.3). Set $l_k = n_k - n_1$. By definition, $x_{-n_1} = f^{l_k}(x_{-n_k})$. The heights of S^k over the points $\hat{f}^{-n_k}(\hat{x})$ tend to minus infinity. Indeed, they are equal to $\ln|(f^{-l_k})'(x_{-n_1})|$ plus the height of S^1 over $\hat{f}^{-n_1}(\hat{x})$ (with respect to the standard Euclidean metric in a chart near x_{-n_1}). The latter logarithm tends to $-\infty$, since $(f^{-l_k})'(x_{-n_1}) \to 0$ (the Shrinking Lemma). Thus, $h_k(x_{-n_k}) \to -\infty$, $x_{-n_k} \to b = 0$, by construction. This together with Proposition 3.4 proves (3.3) and finishes the proof of Theorem 1.53 in the case, when $L \subset \mathcal{A}_f^l$. □

3.3 Case when $L \subset \mathcal{A}_f \setminus \mathcal{A}_f^l$

In this case the function $p : L \to p(L) \subset \mathcal{A}_f^n$ is not an affine isomorphism. Lemma 3.3 and hence, the above construction do not apply literally, and we have to modify our arguments. We fix a $\hat{x} \in L$, $\pi_0(\hat{x}) \in J$, and a sequence $n_k \in \mathbb{N}$, set

$$\hat{y} = p(\hat{x}) \in \mathcal{A}_f^n = \mathcal{A}_f^l, \text{ such that } n_k \to \infty, \ y_{-n_k} \to b \in \overline{\mathbb{C}}, \text{ as } k \to \infty,$$

b is not a parabolic periodic point. The existence of these n_k and b follows from the condition of Theorem 1.53, which says that the projection $p(L) \subset \mathcal{A}_f^n$ is not contained in a leaf associated to a parabolic periodic point. For any critical point c of f denote d_c the local degree of f at c: the multiplicity of c as a preimage of its critical value. Set

$$d = \prod_{c \in J} d_c : \text{ the product being taken through all the critical points of } f \text{ in } J. \tag{3.7}$$

We show that passing to a subsequence, one can achieve that there exists a disk $U \subset \overline{\mathbb{C}}$ centered at b such that each uniformized local leaf $L(\hat{f}^{-n_k}(\hat{x}), U)$ (see the next definition) is a branched cover over U of one and the same degree $\nu \leq d$, and each local leaf $L(\hat{f}^{-n_k}(\hat{y}), U)$ is univalent (Lemmas 3.3, 3.7 and Corollary 3.8). We prove that the ramification points of the above uniformized local leaves tend to b (Proposition 3.9, which provides a stronger statement). We parametrize each uniformized local leaf by unit disk D_1 equipped with the standard Euclidean metric. We show that the corresponding heights of the uniformized horospheres S^k tend to $-\infty$ uniformly on compact sets in D_1 (Proposition 3.10). Afterwards we fix a disk V such that $\overline{V} \subset U \setminus b$, and for any k large enough we fix a local leaf $L_k \subset L(\hat{f}^{-n_k}(\hat{x}), U)$ over V. The convergence of the ramification points implies the univalence of L_k for large k. The height convergence statement (3.3) for thus constructed L_k will be then deduced from Propositions 3.9 and 3.10.

Definition 3.5 Let $L \subset \mathcal{A}_f$ be a leaf, $B \subset L$ be an open domain. Recall that L is isomorphic to either $\mathbb{C}$, or its quotient by a discrete group of Euclidean isometries. Let $\Pi : \mathbb{C} \to L$ be the quotient projection, $\widetilde{B} \subset \mathbb{C}$ be a connected component of $\Pi^{-1}(B)$. The domain $\widetilde{B}$ and the projection $\Pi : \widetilde{B} \to B$ are called the *affine uniformization* of B. If B is a (local) leaf, then $\widetilde{B}$ is called the *uniformized (local) leaf*, and the composition $\pi_0 \circ \Pi : \widetilde{B} \to \pi_0(B)$ is called the *uniformizing (local) leaf projection*. Let H be the hyperbolic leaf in $\mathcal{H}_f$ corresponding to L. The projection Π extends up to a quotient projection $\Pi : \mathbb{H}^3 \to H$ called the uniformization of H. For any horosphere $S \subset H$ its preimage $\widetilde{S} = \Pi^{-1}(S) \subset \mathbb{H}^3$ is a horosphere called the *uniformizing horosphere* of S.

Remark 3.6 Let $L \subset \mathcal{A}_f$ be a leaf, $\hat{x} \in L$,

$$\Phi_{\hat{x}}(t) = (\phi_{0,\hat{x}}, \phi_{-1,\hat{x}}, \dots)(t), \ \Phi_{\hat{x}}(0) = p(\hat{x}) \in \mathcal{A}_f^n : \ \phi_{-j,\hat{x}}(0) = \pi_{-j}(\hat{x}), \tag{3.8}$$

be the meromorphic function sequence (1.4) defining $\hat{x} \in \mathcal{A}_f$. For each $w \in \mathbb{C}$ the meromorphic function sequence $\Phi_{\hat{x}}(w + t)$ represents a point of L denoted $\hat{w}$. The mapping $\mathbb{C} \to L$: $w \mapsto \hat{w}$ is an affine uniformization, and $\phi_{0,\hat{x}} : \mathbb{C} \to \overline{\mathbb{C}}$ is the uniformizing projection. A local leaf is univalent, if and only if so is the uniformizing local leaf projection.

Lemma 3.7 *Let f be a critically-nonrecurrent rational function, d be as in (3.7). For any $b \in J = J(f)$ that is not a parabolic periodic point there exists a disk $U \subset \mathbb{C}$ centered at b such that for any $\hat{b} \in \mathcal{A}_f$ with $\pi_0(\hat{b}) = b$ the corresponding uniformized local leaf over U is simply connected and is a branched cover over U of degree no greater than d.*

Proof By critical nonrecurrence, there exists a $\varepsilon > 0$ (let us fix it) such that

$$dist(f^n(c), c) > \varepsilon \text{ for any critical point } c \in J = J(f) \text{ of } f,\ n \in \mathbb{N}, \tag{3.9}$$

$$dist(c, c') > \varepsilon \text{ for any two distinct (may be multiple) critical points } c, c' \in J. \tag{3.10}$$

Fix an arbitrary disk $U \subset \overline{\mathbb{C}}$ centered at b and disjoint from the forward orbits of the critical points in the Fatou set, such that for any $n \in \mathbb{N} \cup 0$ each connected component of the preimage $f^{-n}(U)$ has diameter less than ε (Mañe's Theorem). Let us prove the lemma for this U. To do this, we use the following

Claim 1. *Let $V, W \subset \overline{\mathbb{C}}$ be connected domains, W be homeomorphic to disk. Let $f : V \to W$ be a finite degree branched covering with a unique critical point c. Let d be the local degree of c. Then V is simply connected and $deg f = d$.*

Claim 1 follows immediately from the Riemann-Hurwitz Formula.

Case 1): $\hat{b} \in \mathcal{A}_f^l$. For any n let U_{-n} be the above connected component that contains b_{-n}. Each U_{-n} contains at most one (may be multiple) critical point of f, by the inequalities $diam(U_{-n}) < \varepsilon$ and (3.10). A critical point c cannot be contained in two domains U_{-n} and U_{-n-m}: otherwise $c, f^m(c) \in U_{-n}$, which is impossible by the diameter bound and (3.9). Thus, each mapping $f : U_{-n} \to U_{-n+1}$ is either a conformal diffeomorphism, or a branched covering with a unique critical point $c_n \in J$. The domains U_n are simply connected, and the degrees of the latter coverings are equal respectively to 1 and d_{c_n} (Claim 1 and induction in n). The number of the values of n corresponding to branchings is no greater than the number of the critical points in the Julia set. Let m be the maximal one of these n. Then $\pi_{-m} : L(\hat{b}, U) \to U_{-m}$ is a diffeomorphism (recall that $\hat{b} \in \mathcal{A}_f^l$ by assumption). Hence, the local leaf $L(\hat{b}, U)$ is simply connected, as is U_{-m}, and $\pi_0 = f^m \circ \pi_{-m} : L(\hat{b}, U) \to U$ is a branched cover of degree $\prod d_{c_n} \leq d$. The local leaf $L(\hat{b}, U)$ is isomorphic to its affine uniformization, since it is contained in $\mathcal{A}_f^l$. This proves the statement of the lemma for $\hat{b} \in \mathcal{A}_f^l$.

Case 2): $\hat{b} \in \mathcal{A}_f \setminus \mathcal{A}_f^l$. The set $\mathcal{A}_f^l$ is dense in $\mathcal{A}_f$ (Proposition 1.38). Fix a sequence $\hat{b}^n \in \mathcal{A}_f^l$, $b_0^n = b$, $\hat{b}^n \to \hat{b}$ in $\mathcal{A}_f$. Let $\phi_{0,\hat{b}^n}$, $\phi_{0,\hat{b}}$ be the corresponding functions from (3.8) normalized so that $\phi_{0,\hat{b}^n} \to \phi_{0,\hat{b}}$ uniformly on compact sets in $\mathbb{C}$. Recall that their values at 0 equal b. Set $B^n = L(\hat{b}^n, U)$. Let $\widetilde{B}^n$ ($\widetilde{B}$) denote the connected component containing 0 of the preimage $\phi_{0,\hat{b}^n}^{-1}(U)$ (respectively, $\phi_{0,\hat{b}}^{-1}(U)$). Then each mapping $\phi_{0,\hat{b}^n} : \widetilde{B}^n \to U$ is the uniformizing projection corresponding to the local leaf B^n. Hence, it is a branched cover of degree at most d, and $\widetilde{B}^n$ is simply connected, as was proved above.

Claim 2. *The domain $\widetilde{B}$ is simply connected, and the mapping $\phi_{0,\hat{b}} : \widetilde{B} \to U$ is a branched cover of degree at most d.*

Proof The simply connected domains $\widetilde{B}^n$ are conformally equivalent to the unit disk D_1, since $\phi_{0,\hat{b}^n} : \widetilde{B}^n \to U$ are nonconstant bounded holomorphic functions. Fix some conformal isomorphisms $\eta_n : D_1 \to \widetilde{B}^n$, $\eta_n(0) = 0$. Passing to a subsequence, one can achieve that η_n

converge to a nonconstant univalent function $\eta : D_1 \to \mathbb{C}$. This follows from the compactness of the space of normalized univalent functions, Köbe $\frac{1}{4}$ theorem and the uniform boundedness of the distances $dist(\partial\widetilde{B}^n, 0)$ from above and from below. Indeed, $dist(\partial\widetilde{B}^n, 0) \to dist(\partial\widetilde{B}, 0)$, as $n \to \infty$, by the convergence $\phi_{0,\hat{b}^n} \to \phi_{0,\hat{b}}$. The mappings $g_n = \phi_{0,\hat{b}^n} \circ \eta_n : D_1 \to U$ are Bläschke products of degree at most d, since they are branched covers of degree at most d, as are $\phi_{0,\hat{b}^n} : \widetilde{B}^n \to U$, and D_1, U are disks. One has $g_n \to g = \phi_{0,\hat{b}} \circ \eta \not\equiv const$ uniformly on compact sets. The limit g is a Bläschke product of degree at most d, as are g_n. Therefore, the mapping $\phi_{0,\hat{b}} : \eta(D_1) \to U$ is a branched cover of degree at most d. The domain $\eta(D_1)$ is simply connected (univalence) and coincides with $\widetilde{B}$ by construction. The claim is proved. $\Box$

The mapping from Claim 2 is the uniformizing projection corresponding to the local leaf $L(\hat{b}, U)$. This proves the lemma. $\Box$

Corollary 3.8 *Let $\hat{x} \in \mathcal{A}_f$, $\hat{y} = p(\hat{x}) \in \mathcal{A}_f^l$, $y_0 = \pi_0(\hat{x}) \in J$, and $L(\hat{y})$ be not a leaf associated to a parabolic periodic point. Let $b \in J$ be a limit point of the sequence $y_0, y_{-1}, \dots$ that is not a parabolic periodic point. There exist a disk $U \subset \overline{\mathbb{C}}$ centered at b, a $\nu \in \mathbb{N}$, $\nu \leq d$, and a sequence $n_k \in \mathbb{N}$ satisfying the following statements:*

$$y_{-n_k} = \pi_{-n_k}(\hat{x}) \to b, \ \text{as } k \to \infty; \tag{3.11}$$

$$\text{the local leaves } L(\hat{f}^{-n_k}(\hat{y}), U) \text{ are univalent over } U; \tag{3.12}$$

the uniformizing projections corresponding to the local leaves $L(\hat{f}^{-n_k}(\hat{x}), U)$ are branched covers over U of degree ν.

Proof There exist a sequence n_k and a disk U_1 centered at b that satisfy (3.11) and (3.12) (Lemma 3.3). There exists another disk U_2 centered at b and satisfying the statements of Lemma 3.7. Let U be the smallest one of the disks U_1 and U_2. The local leaves $L(\hat{f}^{-n_k}(\hat{y}), U)$ remain univalent after passing to a smaller disk. The uniformized local leaves $L(\hat{f}^{-n_k}(\hat{x}), U)$ remain branched covers over U of degrees $\nu_k \leq d$. Passing to a subsequence one can achieve that the degrees ν_k are the same. This proves the corollary. $\Box$

We fix a $\hat{x} \in L$, set $\hat{y} = p(\hat{x})$, and b, n_k, ν, U, as in Corollary 3.8. Without loss of generality we consider that there is a bigger disk $U' \Supset U$ over which the local leaves $L(\hat{f}^{-n_k}(\hat{y}), U')$ are univalent (shrinking U and passing to a subsequence, as in the proof of the corollary). For any $k \geq 2$ set

$$l_k = n_k - n_1, \ f^{-l_k} : U' \to \overline{\mathbb{C}} \text{ the inverse branches such that } f^{-l_k}(y_{-n_1}) = y_{-n_k}.$$

These branches are holomorphic, by the univalence of the local leaves $L(\hat{f}^{-n_k}(\hat{y}), U')$. By Shrinking Lemma and (3.11), one has

$$f^{-l_k}|_U \to b \text{ uniformly on } U. \text{ We consider that } f^{-l_k}(U) \subset U \tag{3.13}$$

for all k, passing to an appropriate subsequence. Without loss of generality everywhere below we consider that

$$U = D_1, \ b = 0. \text{ Set } \hat{b}^k = \hat{f}^{-n_k}(\hat{x}).$$

For any k let $\widetilde{B}^k \subset \mathbb{C}$ denote the connected component containing 0 of the preimage $\phi_{0,\hat{b}^k}^{-1}(U)$, i.e., the uniformized local leaf $L(\hat{b}^k, U)$. The domains $\widetilde{B}^k$ are simply connected, and $\phi_{0,\hat{b}^k}$:

$\widetilde{B}^k \to U$ are branched covers of degree ν (Corollary 3.8). Fix conformal isomorphisms $\eta_k : D_1 \to \widetilde{B}^k$, $\eta_k(0) = 0$. Set

$$g_k = \phi_{0,\hat{b}^k} \circ \eta_k : D_1 \to U = D_1, \tag{3.14}$$

which are branched covers of degree ν, and hence, Bläschke products,

$$g_k(0) = 0, \ g_k(1) = 1, \tag{3.15}$$

after appropriate normalization of η_k by rotation in the source.

Proposition 3.9 *One has $g_k(z) \to g(z) = z^\nu$ uniformly on $\overline{D_1}$.*

Proof Set

$$C_k = \{\text{the critical values of } \phi_{0,\hat{b}^k}\} = \{\text{the critical values of } g_k\} \subset U = D_1.$$

Each C_k is a set of $\nu - 1$ points, some of them may coincide. One has

$$C_k \to b = 0, \ \text{as } k \to \infty, \tag{3.16}$$

by the convergence $f^{-l_k}|_U \to 0$ and since $f^{-l_k}(C_1) = C_k$. Indeed, $f^{-l_k}(C_1) \subset C_k$, since $\phi_{0,\hat{b}^k} = f^{-l_k} \circ \phi_{0,\hat{b}^1}$ up to $\mathbb{C}^*$- action in the source, f^{-l_k} is holomorphic on U and $f^{-l_k}(U) \subset U$, see (3.13). Therefore, $f^{-l_k}(C_1) = C_k$, since the total multiplicities of the critical values in C_1 and C_k are equal. Passing to a subsequence one can achieve that g_k converge uniformly on compact sets and $C_k \subset D_{\frac{1}{2}}$. Let g denote the limit of g_k. Let us show that $g(z) = z^\nu$. To do this, we consider the auxiliary annulus

$$A = D_1 \setminus \overline{D_{\frac{1}{2}}}, \ \text{set } \widetilde{A}_k = g_k^{-1}(A), \ \widetilde{A} = g^{-1}(A).$$

One has $C_k \cap A = \emptyset$ for all k. Hence, each mapping $g_k : \widetilde{A}_k \to A$ is a nonramified covering of degree ν, and $\widetilde{A}_k$ is an annulus adjacent to ∂D_1, $mod(\widetilde{A}_k) = \frac{1}{\nu} mod(A)$. Set

$$W_k = D_1 \setminus \widetilde{A}_k = g_k^{-1}(\overline{D_{\frac{1}{2}}}). \tag{3.17}$$

There exists a $0 < r < 1$ such that

$$W_k \Subset D_r \text{ for all } k. \tag{3.18}$$

Indeed, $0 \in g_k^{-1}(0) \subset W_k$, see (3.17). Therefore, the contrary to (3.18) would imply that the complement W_k to $\widetilde{A}_k$ contains both the origin and points arbitrarily close to ∂D_1. Hence, the moduli $mod(\widetilde{A}_k)$ are arbitrarily small, - a contradiction to the equality $mod(\widetilde{A}_k) = \frac{1}{\nu} mod(A)$.

The limit g is not a constant. Indeed, otherwise, $g_k \to 0 = g_k(0)$ and $g_k(D_r) \subset D_{\frac{1}{2}}$ for large k, hence, $W_k \supset D_r$, see (3.17), - a contradiction to (3.18). Therefore, g is a Bläschke product, as are g_k. The critical points of g_k lie in $W_k \subset D_r$, since $C_k \subset D_{\frac{1}{2}}$ and by (3.17). Therefore, passing to a subsequence one can achieve that they converge, and their limits are critical points of g. Thus, g is a branched cover of degree ν with a unique critical value 0, see (3.16). In particular, the convergence $g_k \to g$ is uniform on $\overline{D_1}$, since the latter is a convergence of Bläschke products of a given degree to a Bläschke product of the same

degree. The preimage $g^{-1}(D_1 \setminus 0)$ is a finite regular cover over a punctured disk, and hence, conformally equivalent to a punctured disk. It is adjacent to ∂D_1 and does not contain 0. Hence, it is $D_1 \setminus 0$, $g^{-1}(0) = 0$ and $g(z) = cz^{\nu}$. Now $c = 1$, by (3.15). This proves the proposition. □

Fix a horosphere $S \subset H$. Set

$$S^k = \hat{f}^{-n_k}(S), \ \widetilde{S}_k = \text{ the uniformizing horosphere of } S^k,$$

see Definition 3.5. Consider the pushforward under $\eta_k : D_1 \to \widetilde{B}^k$ of the standard Euclidean metric on D_1. This is an Euclidean metric on $\widetilde{B}^k$. For any $w \in D_1$ set

$$\widetilde{h}_k(w) = \text{ the height of } \widetilde{S}_k \text{ over the point } \eta_k(w) \text{ in the above metric.}$$

Proposition 3.10 *One has $\widetilde{h}_k \to -\infty$ uniformly on compact sets in D_1, as $k \to \infty$.*

Proof The mapping $\hat{f}^{-l_k}$ sends the local leaf $L(\hat{f}^{-n_1}(\hat{x}), U)$ to $L(\hat{f}^{-n_k}(\hat{x}), U)$, by definition and (3.13). Its lifting to the affine uniformization is an affine mapping $\widetilde{B}^1 \to \widetilde{B}^k$, which will be also denoted $\hat{f}^{-l_k}$. Set

$$\chi_k = \eta_k^{-1} \circ \hat{f}^{-l_k} \circ \eta_1 : D_1 \to D_1. \text{ One has}$$

$$\widetilde{h}_k(\chi_k(w)) = \widetilde{h}_1(w) + \log|\chi_k'(w)|, \tag{3.19}$$

$$\chi_k \to 0 \text{ uniformly on compact sets in } D_1, \text{ as } k \to \infty, \tag{3.20}$$

by definition, (3.13), Proposition 3.9 and since χ_k coincides with an analytic branch of the composition $g_k^{-1} \circ f^{-l_k} \circ g_1 : D_1 \to D_1$. Hence, $\chi_k' \to 0$. This together with (3.19) implies that $\widetilde{h}_k(\chi_k(0)) \to -\infty$. Hence, $\widetilde{h}_k \to -\infty$ uniformly on compact sets, by (3.20) and equicontinuity (Proposition 3.4). This proves Proposition 3.10. □

Proof of Theorem 1.53. Let $\hat{x}$, n_k, U, b be the same, as in Corollary 3.8. Fix an arbitrary disk V such that $\overline{V} \subset U \setminus b$. The uniformizing projections corresponding to the local leaves $L(\hat{f}^{-n_k}(\hat{x}), U)$ are branched covers over U. Without loss of generality we consider that $\overline{V}$ contains no their critical values, passing to k large enough, see (3.16). For any k fix a local leaf $L_k \subset L(\hat{f}^{-n_k}(\hat{x}), U)$ over V. It is univalent by construction. This proves statement (3.1). Let us prove (3.3). Its stronger version says that the heights of the horospheres $\widetilde{S}_k$ over L_k in the standard Euclidean metric of the disk V tend to $-\infty$ uniformly on V. The latter heights differ from the heights $\widetilde{h}_k$ by the logarithm of the ratio of the standard Euclidean metric on the disk U and the pushforward under g_k of the Euclidean metric on D_1. The latter ratio is uniformly bounded from above and from below on $\overline{V}$, and there exists a $0 < r < 1$ such that $g_k^{-1}(\overline{V}) \subset \overline{D}_r$ for all k (Proposition 3.9). This together with Proposition 3.10 proves (3.3). Thus, statements (3.1) and (3.3) are proved for the above L_k and V. This together with the discussion in 3.1 proves Theorem 1.53. □

4 Closeness of the horospheres associated to the parabolic periodic points

Here we prove Theorem 1.54. Let f be a rational function with a parabolic periodic point $a \in \overline{\mathbb{C}}$. Without loss of generality we consider that a is fixed, passing to appropriate iteration

of f. Let $L_a \subset \mathcal{A}_f$ be the leaf associated to a of the affine lamination, $H_a \subset \mathcal{H}_f$ be the corresponding hyperbolic leaf. In the proof of Theorem 1.54 we use the following proposition.

Proposition 4.1 *Let f, a, H_a be as above. Then each horosphere in H_a is invariant under the mapping $\hat{f}$.*

Proof The Fatou coordinate is affine on the leaf L_a, and $\hat{f}$ acts by unit translation there. Hence, it preserves a Euclidean metric on L_a. Therefore, the lifting of $\hat{f}$ to H_a preserves the corresponding height. This implies the proposition. □

Fix a horosphere $S \subset H_a$. We show that S is closed in $\mathcal{H}_f$ and does not accumulate to itself. This together with its invariance (Proposition 4.1) implies Theorem 1.54.

We prove the previous statement by contradiction. Suppose the contrary: then S accumulates to a horosphere $S' \subset \mathcal{H}_f$. Let $H \subset \mathcal{H}_f$ be the leaf containing S', $L \subset \mathcal{A}_f$ be the corresponding affine leaf. Fix a disk $U \subset \overline{\mathbb{C}}$, a $(\hat{b}, h) \in S'$, a sequence $(\hat{b}^k, h_k) \in S$, $\hat{b} \in L$, $\hat{b}^k \in L_a$, such that

$$\pi_0(\hat{b}) \in U, \pi_0(\hat{b}) \neq a, \text{ and the local leaf } L(\hat{b}, U) \text{ is univalent over } U, \tag{4.1}$$

$$(\hat{b}^k, h_k) \to (\hat{b}, h) \text{ as } k \to \infty, \ \pi_0(\hat{b}^k) = \pi_0(\hat{b}), \ \hat{b}^k \neq \hat{b} \text{ for all } k. \tag{4.2}$$

Here the heights of the horospheres are measured with respect to the Euclidean metric of U. Without loss of generality we consider that $U = D_1$ and $\pi_0(\hat{b}) = 0$. Fix an arbitrary disk V centered at 0 such that $\overline{V} \subset U$. Then for any k large enough the local leaf

$$\Lambda_k = L(\hat{b}^k, V) \text{ is univalent over } V,$$

by (4.1) and the definition of topology on $\mathcal{A}_f$. Without loss of generality we consider that all $\hat{b}^k$, and hence, Λ_k, are distinct. One can achieve this by passing to a subsequence, by (4.2). We show that

$$h_k \to +\infty, \text{ as } k \to \infty, \tag{4.3}$$

- a contradiction to (4.2). This proves Theorem 1.54.

The inverse germ f^{-1} fixing a will be called the *parabolic germ*. For the proof of (4.3) we fix a $r > 0$, set $\Delta = D_{2r}(a)$, such that

$$\text{the parabolic inverse germ } f^{-1} \text{ is holomorphic on } \Delta \cup f(\Delta) \text{ and } f(\overline{D_r(a)}) \subset \Delta; \tag{4.4}$$

$$\text{if } z \in \overline{D_r(a)}, \text{ then the parabolic branch } f^{-1}|_\Delta \text{ sends } f(z) \in \Delta \text{ to } z; \tag{4.5}$$

$$\text{each infinite backward orbit of } f \text{ contained in } \overline{D_r(a)} \text{ converges to } a. \tag{4.6}$$

Statements (4.4)-(4.6) hold true, whenever r is small enough. For any k let $n_k \in \mathbb{N}$ be the minimal number such that $b^k_{-j} \in \overline{D_r(a)}$, whenever $j \geq n_k$. The number n_k exists, since $\hat{b}^k \in L_a$ and by Proposition 1.27. Set

$$\hat{v}^k = \hat{f}^{-n_k}(\hat{b}^k).$$

Passing to a subsequence, one can achieve that $v^k_0 = b^k_{-n_k}$ converge to some point $x_0 \in \overline{D_r(a)}$. Then $v^k_{-j} \to x_{-j} \in \overline{D_r(a)}$ for all j, and both $\hat{v}^k$ and $\hat{x} = (x_0, x_{-1}, \dots)$ are infinite backward orbits of f in $\overline{D_r(a)}$, by (4.5). They converge to a by (4.6). They are distinct from the fixed

orbit of a. For the former, $\hat{v}^k$, this follows from definition and the inequality $b_0^k = \pi_0(\hat{b}) \neq a$, see (4.1). For the latter, $\hat{x}$, this holds true, since $x_0 = \lim_{k\to\infty} b^k_{-n_k}$ and $b^k_{-n_k}$ are bounded away from a: either $n_k = 0$ and $b^k_{-n_k} = \pi_0(\hat{b}) \neq a$, or $b^k_{-n_k+1} = f(b^k_{-n_k}) \notin D_r(a)$, by definition. The backward iterations $f^{-j} : x_0 \mapsto x_{-j}$ are powers of the parabolic inverse branch, by (4.5). They are holomorphic in a neighborhood $W = W(x_0)$ and converge to a uniformly on compact subsets in W, since the attractive basin of a parabolic fixed point is open. Fix this W. Then the local leaf $L(\hat{x}, W)$ is univalent over W, and $\hat{v}^k \to \hat{x}$ along this local leaf, by definition and (4.5). In particular, $L(\hat{x}, W) \subset L_a$, since $\hat{v}^k \in L_a$.

The sequence n_k tends to infinity, by definition and since the local leaves Λ_k are distinct. Fix a metric on W. The corresponding height of $S = \hat{f}^{-n_k}(S)$ over $\hat{v}^k$ tends to a finite value, namely, to its height over $\hat{x}$, as $k \to \infty$ (since $\hat{v}^k \to \hat{x}$ along a fixed local leaf). On the other hand, its difference with the height h_k of S over $\hat{b}^k$ is equal to $\ln |(f^{-n_k})'(0)|$, which tends to $-\infty$ (by the Shrinking Lemma). This implies (4.3) and proves Theorem 1.54.

5 Nondense horospheres. Proof of Theorem 1.57

Here we give a proof of Theorem 1.57 (Subsections 5.1 – 5.4). Its Addendum is proved analogously with small modifications discussed at the end of the paper.

5.1 The plan of the proof of Theorem 1.57

It suffices to prove Theorem 1.57 for $h = 0$, as in the proof of Theorem 1.49 in Section 2.

Let $\varepsilon < \frac{1}{4}$, $\varepsilon \neq 0, -2$. Let $a = a(\varepsilon)$ be the fixed point (1.19) of f_ε, $\hat{a} \in \mathcal{A}_{f_\varepsilon}$ be its fixed orbit, $\Pi_a \subset \mathcal{A}_{f_\varepsilon}$ be the corresponding subset from (2.3). The set Π_a is nonempty, since the fixed point $a(\varepsilon)$ is not branch-exceptional (Propositions 1.56 and 2.1). We prove Theorem 1.57 for

$$B_\varepsilon = \overline{\{\beta(\hat{a}, \hat{y}) \mid \hat{y} \in \Pi_a\}}. \text{ Set } SB_\varepsilon = \cup_{\beta \in B_\varepsilon} S_{\hat{a},\beta} \subset \mathcal{H}_{f_\varepsilon}. \tag{5.1}$$

To do this, we show that for every $\varepsilon < \frac{1}{4}$, $\varepsilon \neq 0, -2$,

$$B_\varepsilon = \{\sum_{m=1}^{l} \beta(\hat{a}, \hat{y}(m)) \mid l \in \mathbb{N},\ \hat{y}(m) \in \Pi_a\}; \tag{5.2}$$

$$S_{\hat{a},0} \text{ accumulates exactly to } SB_\varepsilon. \tag{5.3}$$

Statement (5.3) is proved in 5.4. Statements (5.2) is proved below and in 5.2, 5.3. In the proof of (5.2) we use the well-known formula for the basic cocycle $\beta(\hat{x}, \hat{y})$ as an infinite series in $\ln |f'_\varepsilon(x_{-j})|$ and $\ln |f'_\varepsilon(y_{-j})|$ (the next proposition and (5.5)). We show (Lemma 5.2 and Corollary 5.3) that if $\varepsilon < 0$ ($0 < \varepsilon < \frac{1}{4}$), then the module $|f'_\varepsilon|$ achieves its maximum (minimum) on the Julia set exactly at $\pm a(\varepsilon)$. This together with (5.5) implies that $\pm\beta(\hat{a}, \hat{y}) > 0$ whenever $\pm\varepsilon > 0$ and $\hat{y} \in \Pi_a$ (Corollary 5.4). (Moreover, for a given ε, the values $\beta(\hat{a}, \hat{y})$ are bounded away from 0.) Using this statement, we show (Lemma 5.5 and Corollary 5.6) that for every sequence $\hat{a}^n \in \Pi_a$ with converging values $\beta(\hat{a}, \hat{a}^n)$ the limit of the latter is a sum (5.2). This implies (5.2).

Proposition 5.1 *Let $f(z)$ be a rational function, $\hat{x}, \hat{y} \in \mathcal{A}^l_f$ be a pair of points lying on one and the same leaf, $x_{-j}, y_{-j} \in \mathbb{C}$ for all j (the equality $x_0 = y_0$ does not necessarily hold). Let*

$\beta_{\hat{x}}$, $\beta_{\hat{y}}$ be the height functions (1.12) on the corresponding hyperbolic leaf in $\mathcal{H}_f$ defined by the standard Euclidean metric on $\mathbb{C}$. Then

$$\beta(\hat{x},\hat{y}) = \beta_{\hat{y}} - \beta_{\hat{x}} = \sum_{j=1}^{+\infty}(\ln|f'(y_{-j})| - \ln|f'(x_{-j})|). \tag{5.4}$$

The proposition follows from formula (3.23) in [11]. It implies that for every $\hat{y} \in \Pi_a$

$$\beta(\hat{a},\hat{y}) = \sum_{j=1}^{+\infty}(\ln|f'_\varepsilon(y_{-j})| - \ln|f'_\varepsilon(a(\varepsilon))|) \tag{5.5}$$

Lemma 5.2 *The Julia set $J = J(f_\varepsilon)$ is*
- *contained in $D_{a(\varepsilon)} \cup \pm a(\varepsilon)$, $D_{a(\varepsilon)} = \{|z| < a(\varepsilon)\}$, whenever $\varepsilon < 0$;*
- *contained in the complement $\mathbb{C} \setminus (\overline{D_{a(\varepsilon)}} \setminus \pm a(\varepsilon))$, if $0 < \varepsilon < \frac{1}{4}$.*

Lemma 5.2 is proved in the next subsection.

Corollary 5.3 *If $\varepsilon < 0$ ($0 < \varepsilon < \frac{1}{4}$), then for every $x \in J(f_\varepsilon)$ one has $|f'_\varepsilon(x)| \leq f'_\varepsilon(a(\varepsilon))$ (respectively, $|f'_\varepsilon(x)| \geq f'_\varepsilon(a(\varepsilon))$). In both cases the equality is achieved exactly at $x = \pm a(\varepsilon)$.*

Proof One has $f'_\varepsilon(x) = 2x$. This together with the lemma implies the corollary. □

Corollary 5.4 *If $\varepsilon \in (-\infty, 0) \setminus \{-2\}$ ($0 < \varepsilon < \frac{1}{4}$), then for every $\hat{y} \in \Pi_a$ each corresponding nonzero term of the sum in (5.5), and hence, $\beta(\hat{a},\hat{y})$, is negative (respectively, positive). The zero terms correspond exactly to $y_{-j} = \pm a(\varepsilon)$.*

Lemma 5.5 *Let $\varepsilon \in (-\infty, \frac{1}{4}) \setminus \{-2, 0\}$. Let $\hat{a}^n \in \Pi_a$ be a sequence of points such that the values $\beta(\hat{a},\hat{a}^n)$ converge to a finite limit and $a^n_{-1} = -a(\varepsilon)$. Then (passing to a subsequence one can achieve that) there exists a $l \in \mathbb{N}$ such that for every n there exist indices*

$$0 = \nu_1 < \nu_2 < \dots < \nu_l, \ \nu_m = \nu_m(\hat{a}^n), \text{ we set } \nu_{l+1} = +\infty,$$

such that each $m = 1, \dots, l$ satisfies the following statements:

$$a^n_{-\nu_m(\hat{a}^n)} \to a(\varepsilon), \text{ as } n \to \infty, \tag{5.6}$$

$$\nu_{m+1}(\hat{a}^n) - \nu_m(\hat{a}^n) \to +\infty, \tag{5.7}$$

$$\text{for every } j \in \mathbb{N} \text{ the sequence } a^n_{-\nu_m(\hat{a}^n)-j} \text{ converges; set } y_{-j}(m) = \lim_{n\to\infty} a^n_{-\nu_m(\hat{a}^n)-j}, \tag{5.8}$$

$$\hat{y}(m) = (y_0(m), y_{-1}(m), \dots) \text{ represents a point from } \Pi_a, \tag{5.9}$$

$$\Sigma_m = \sum_{j=\nu_m(\hat{a}^n)+1}^{\nu_{m+1}(\hat{a}^n)} (\ln|f'_\varepsilon(a^n_{-j})| - \ln|f'_\varepsilon(a(\varepsilon))|) \to \beta(\hat{a},\hat{y}(m)), \text{ as } n \to \infty. \tag{5.10}$$

Corollary 5.6 *In Lemma 5.5 one has $\beta(\hat{a},\hat{a}^n) \to \sum_{m=1}^{l} \beta(\hat{a},\hat{y}(m))$, as $n \to \infty$.*

The corollary follows from (5.5) and (5.10).

Proof of Theorem 1.57 modulo Lemmas 5.2, 5.5 and (5.3). Consider an arbitrary converging sequence of basic cocycle values $\beta(\hat{a}, \hat{a}^n)$, $\hat{a}^n \in \Pi_a$. Without loss of generality we consider that $a_{-1}^n \neq a(\varepsilon)$ for all n; then $a_{-1}^n = -a(\varepsilon)$, since $f_\varepsilon^{-1}(a(\varepsilon)) = \pm a(\varepsilon)$. One can achieve this replacing $\hat{a}^n$ by $\hat{f}^{-u_n}(\hat{a}^n)$, $u_n = \max\{j \mid a_{-j}^n = a(\varepsilon)\}$. The values $\beta(\hat{a}, \hat{a}^n)$ remain unchanged, by the invariance of basic cocycle. Then $\hat{a}^n$ satisfy the conditions of Lemma 5.5. Therefore, $\beta(\hat{a}, \hat{a}^n)$ converge to a sum (5.2) (Corollary 5.6). This together with the semigroup property (Corollary 2.4) proves (5.2). The set B_ε is a countable subset in $\mathbb{R}_\pm$, by (5.2) and the countability of the set Π_a. This together with (5.3) proves Theorem 1.57. □

5.2 The disk containing the Julia set. Proof of Lemma 5.2

We prove Lemma 5.2 in the case, when $0 < \varepsilon < \frac{1}{4}$. Its proof for $\varepsilon < 0$ is analogous and will be discussed at the end of the subsection.

Let $0 < \varepsilon < \frac{1}{4}$. We have to show that $J \cap \overline{D_{a(\varepsilon)}} = \pm a(\varepsilon)$. To do this, we prove that

$$|f_\varepsilon(x)| < a(\varepsilon) \text{ for every } x \in \partial D_{a(\varepsilon)} \setminus \pm a(\varepsilon). \tag{5.11}$$

This implies that f_ε maps the disk $D_{a(\varepsilon)}$ to itself, by the maximum principle. Hence, this disk does not intersect the Julia set, by Montel's theorem ([12], p.52). One has $f_\varepsilon(\overline{D_{a(\varepsilon)}} \setminus \pm a(\varepsilon)) \subset D_{a(\varepsilon)}$, by (5.11). Thus, the set $\overline{D_{a(\varepsilon)}} \setminus \pm a(\varepsilon)$ is disjoint from J, as is $D_{a(\varepsilon)}$, by the above inclusion and (1.2). The points $\pm a(\varepsilon)$ belong to the Julia set, since they are mapped to the repelling fixed point $a(\varepsilon)$. This proves the second statement of the lemma modulo (5.11).

Proof of (5.11). By definition, $f_\varepsilon(a(\varepsilon)) = a^2(\varepsilon) + \varepsilon = a(\varepsilon)$. Hence, for every $\phi \in \mathbb{R}$ one has

$$f_\varepsilon(a(\varepsilon)e^{i\phi}) = a^2(\varepsilon)e^{2i\phi} + \varepsilon = e^{2i\phi}(a^2(\varepsilon) + \varepsilon - \varepsilon(1 - e^{-2i\phi})) = e^{2i\phi}(a(\varepsilon) - \varepsilon(1 - e^{-2i\phi})).$$

Statement (5.11) in a equivalent reformulation says that the latter right-hand side has module less than $a(\varepsilon)$, whenever $e^{2i\phi} \neq 1$. Or equivalently,

$$\Gamma_\varepsilon = \{\varepsilon(1 - e^{2i\phi}) \mid \phi \in \mathbb{R}\} \subset D_{a(\varepsilon)}(a(\varepsilon)) \cup 0. \tag{5.12}$$

Indeed, the circle Γ_ε is centered at $\varepsilon > 0$, has radius ε and is tangent to the boundary $\partial D_{a(\varepsilon)}(a(\varepsilon))$ at 0. One has $a(\varepsilon) > \varepsilon$: the forward orbit of the critical value ε converges to a finite attracting fixed point ([12], p.62), while the orbit of every $x > a(\varepsilon)$ tends to infinity. The two latter statements imply (5.12). This proves (5.11) and the second statement of Lemma 5.2. □

Let us now consider the case, when $\varepsilon < 0$. We have to show that $J \subset D_{a(\varepsilon)} \cup \pm a(\varepsilon)$, or equivalently, $\mathbb{C} \setminus (D_{a(\varepsilon)} \cup \pm a(\varepsilon)) \subset \mathbb{C} \setminus J$. To do this, we prove that

$$|f_\varepsilon(x)| > a(\varepsilon), \text{ whenever } x \in \partial D_{a(\varepsilon)} \setminus \pm a(\varepsilon). \tag{5.13}$$

This is proved analogously to (5.11) (now Γ_ε lies outside $D_{a(\varepsilon)}(a(\varepsilon))$, since $\varepsilon < 0$). Then

$$|f_\varepsilon||_{\mathbb{C} \setminus (D_{a(\varepsilon)} \cup \pm a(\varepsilon))} > a(\varepsilon). \tag{5.14}$$

Indeed, the polynomial f_ε has no zeros in the complement $\mathbb{C}\setminus\overline{D_{a(\varepsilon)}}$: both its roots $\pm\sqrt{|\varepsilon|}$ have module less than $a(\varepsilon)$ by (1.19). This together with (5.13) and the maximum principle applied to the function $\frac{1}{f_\varepsilon}(\frac{1}{w})$ implies (5.14). The complement $\mathbb{C}\setminus(D_{a(\varepsilon)}\cup\pm a(\varepsilon))$ is disjoint from the Julia set, by (5.14) and Montel's theorem, as in the previous case. This proves Lemma 5.2.

5.3 Limits of basic cocycles. Proof of Lemma 5.5

Set $\nu_1=0$. The indices $\nu_m(\hat{a}^n)$ with $m\geq 2$ are defined below. They are exactly those indices, for which $a^n_{-\nu_m}$ is close to $a(\varepsilon)$ (as n is big), and $a^n_{-\nu_m-1}$ is obtained from it by the inverse branch f_ε^{-1} sending $a(\varepsilon)$ to $-a(\varepsilon)$. We show that the number of those ν_m is finite and uniformly bounded and prove Lemma 5.5 for them. To do this, we choose appropriate $\sigma>0$ small enough, and for each $\hat{y}\in\Pi_a$ with $y_{-1}=-a(\varepsilon)$ we consider those indices j for which $y_{-j}\in D_\sigma(a(\varepsilon))$ and $y_{-j-1}\notin D_\sigma(a(\varepsilon))$. We show, see (5.28), that their number is bounded from above by a constant depending only on ε, σ and $\beta(\hat{a},\hat{y})$.

Fix a $\varepsilon\in(-\infty,\frac{1}{4})\setminus\{-2,0\}$. Fix a $\sigma>0$ and set

$$D'_\sigma=\text{ the connected component containing }-a(\varepsilon)\text{ of }f_\varepsilon^{-1}(D_\sigma(a(\varepsilon))).\tag{5.15}$$

We choose σ small enough so that

$$f_\varepsilon|_{D_\sigma(a(\varepsilon))}\text{ is univalent and }f_\varepsilon(D_\sigma(a(\varepsilon)))\supset\overline{D_\sigma(a(\varepsilon))},\tag{5.16}$$

$$\text{the open sets }D_\sigma(a(\varepsilon)),\ D'_\sigma,\ f_\varepsilon^{-1}(D'_\sigma),\ f_\varepsilon^{-2}(D'_\sigma)\text{ are disjoint. Set}\tag{5.17}$$

$$\delta=\frac{1}{2}\min\{|\ln|f'_\varepsilon(x)|-\ln|f'_\varepsilon(a(\varepsilon))||\mid x\in J(f_\varepsilon)\setminus(D'_\sigma\cup D_\sigma(a(\varepsilon)))\}.\text{ One has }\delta>0.\tag{5.18}$$

This follows from Corollary 5.3. By (5.16) and (5.17),

$$f_\varepsilon^{-1}(D_\sigma(a(\varepsilon)))=D'_\sigma\bigsqcup D''_\sigma,\ D''_\sigma\subset D_\sigma(a(\varepsilon)).\tag{5.19}$$

Let $\hat{y}\in\Pi_a$, $y_{-1}=-a(\varepsilon)$. Recall that the backward orbit $y_0,y_{-1},\dots$ tends to $a(\varepsilon)$. Therefore, it leaves $D_\sigma(a(\varepsilon))$ only at a finite number of indices, let $s=s(\hat{y})$ denote their number. After each leaving, it returns back either forever, or until the next leaving. Let $J(\hat{y})=\{j_1,\dots,j_s\}$ denote the leaving indices, $K(\hat{y})=\{k_1,\dots,k_s\}$ be the indices of returns:

$$J(\hat{y})=\{j\in\mathbb{N}\cup 0\mid y_{-j}\in D_\sigma(a(\varepsilon)),\ y_{-j-1}\notin D_\sigma(a(\varepsilon))\};\tag{5.20}$$

$$K(\hat{y})=\{j\in\mathbb{N}\mid y_{-j+1}\notin D_\sigma(a(\varepsilon)),\ y_{-j}\in D_\sigma(a(\varepsilon))\}.\text{ One has}$$

$$0=k_0=j_1<k_1\leq j_2<k_2\leq\dots<k_s<j_{s+1}=k_{s+1}=+\infty,\ j_r=j_r(\hat{y}),\ k_r=k_r(\hat{y}),\tag{5.21}$$

$$y_{-j}\notin D_\sigma(a(\varepsilon)),\text{ if and only if }j_r+1\leq j<k_r\text{ for some }r\leq s,\tag{5.22}$$

$$y_{-j}\in D'_\sigma,\text{ if and only if }j=j_r+1\text{ for some }r\leq s,\tag{5.23}$$

$$y_{-j}\notin D_\sigma(a(\varepsilon))\cup D'_\sigma,\text{ if and only if }j_r+2\leq j<k_r\text{ for some }r\leq s,\tag{5.24}$$

$$k_r-j_r\geq 4\text{ for each }r\leq s.\tag{5.25}$$

Statements (5.21) and (5.22) follow from definition. Statement (5.23) follows from definition and (5.19). Statement (5.24) follows from (5.22) and (5.23). One has $y_{-j_r-1}\in D'_\sigma$,

$y_{-j_r-1}, y_{-j_r-2}, y_{-j_r-3} \notin D_\sigma(a(\varepsilon))$ for every $r \leq s$, by (5.17) and (5.23). Hence, $k_r \geq j_r + 4$, by (5.22). This proves (5.25). Set

$$\mathcal{D}(\hat{y}) = \{j \in \mathbb{N} \mid y_{-j} \notin D_\sigma(a(\varepsilon))\}. \text{ By (5.22),}$$

$$\mathcal{D}(\hat{y}) = \{j \in \mathbb{N} \mid j_r(\hat{y}) + 1 \leq j < k_r(\hat{y}) \text{ for some } r \leq s(\hat{y})\}. \text{ Let} \tag{5.26}$$

$$d(\hat{y}) = \#(\mathcal{D}(\hat{y})) \text{ denote its cardinality. One has } s(\hat{y}) \leq d(\hat{y}), \tag{5.27}$$

by (5.25) and (5.26).

The next a priori bound is the main argument in the proof of Lemma 5.5:

$$|\beta(\hat{a}, \hat{y})| > \delta d(\hat{y}) \geq \delta \text{ for every } \hat{y} \in \Pi_a \text{ with } y_{-1} = -a(\varepsilon). \tag{5.28}$$

Proof of (5.28). Each term in (5.5) corresponding to an index j with $y_{-j} \notin D'_\sigma \cup D_\sigma(a(\varepsilon))$ has module no less than 2δ, by the definition of δ. These are exactly the indices from (5.24), and their number equals $\sum_{r=1}^{s}(k_r - j_r - 2)$. All the terms in (5.5) are nonzero and have the same sign, as $\beta(\hat{a}, \hat{y})$, except for zero terms with $j = 0, 1$: $y_{0,-1} = \pm a(\varepsilon)$. This follows from Corollary 5.4. This together with the previous statements and (5.25) implies that

$$\beta(\hat{a}, \hat{y}) \geq 2\delta \sum_{r=1}^{s}(k_r - j_r - 2) > \delta \sum_{r=1}^{s}(k_r - j_r - 1) = \delta d(\hat{y}).$$

□

The numbers $d(\hat{a}^n)$ and $s(\hat{a}^n)$ are uniformly bounded from above, by (5.27), (5.28) and the convergence of the values $\beta(\hat{a}, \hat{a}^n)$. Now passing to a subsequence, one can achieve that

$$s = s(\hat{a}^n), \ d = d(\hat{a}^n) \text{ are independent on } n, \tag{5.29}$$

$$\text{for every fixed } j \in \mathbb{N} \cup 0 \text{ and } r \leq s \text{ the sequence } a^n_{-j_r(\hat{a}^n)-j} \text{ converges, as } n \to \infty. \tag{5.30}$$

Take those values r (denote l their number) for which

$$a^n_{-j_r(\hat{a}^n)} \to a(\varepsilon), \text{ as } n \to \infty; \text{ denote these } r \text{ by } 1 = r(1) < \cdots < r(l). \text{ Set } \nu_{l+1} = +\infty, \tag{5.31}$$

$$\nu_m(\hat{a}^n) = j_{r(m)}(\hat{a}^n), \ y_{-j}(m) = \lim_{n\to\infty} a^n_{-\nu_m(\hat{a}^n)-j} \text{ for every } m = 1, \dots, l, \ j \in \mathbb{N} \cup 0, \tag{5.32}$$

$$\hat{y}(m) = (y_0(m), y_{-1}(m), \dots).$$

Statements (5.6)-(5.8) follow from (5.31) and (5.32).

Proof of (5.9). One has $y_0(m) = a(\varepsilon)$, by (5.31), $y_{-j}(m) \to a(\varepsilon)$, as $j \to +\infty$ (hence, $\hat{y}(m) \in L(\hat{a})$). Indeed, the converse would imply that there exists a $\sigma' > 0$ such that there is an infinite number of indices j for which $y_{-j}(m) \notin \overline{D_{\sigma'}(a(\varepsilon))}$. This together with the convergence $a^n_{-\nu_m(\hat{a}^n)-j} \to y_{-j}(m)$ would imply that there are $\hat{a}^n$ with arbitrarily large number of indices j for which $a^n_{-j} \notin D_{\sigma'}(a(\varepsilon))$. Hence, the number $d(\hat{a}^n)$ constructed above with σ replaced by $\min\{\sigma, \sigma'\}$ is not uniformly bounded, - a contradiction to (5.28) and the convergence of $\beta(\hat{a}, \hat{a}^n)$. Thus, $\hat{y}(m) \in L(\hat{a})$, $y_0(m) = a(\varepsilon)$. One has $\hat{y}(m) \neq \hat{a}$, since $y_{-1}(m) = -a(\varepsilon)$. Indeed, $y_{-1}(m) \in \{\pm a(\varepsilon)\} = f_\varepsilon^{-1}(a(\varepsilon))$, since $y_0(m) = a(\varepsilon)$. By definition, the point $y_{-1}(m)$ is the limit of $a^n_{-1-\nu_m(\hat{a}^n)} \notin D_\sigma(a(\varepsilon))$, see (5.20) and (5.32). Hence, $y_{-1}(m) = -a(\varepsilon)$. The projection $\pi_0|_{L(\hat{a})}$ has nonzero derivative at $\hat{y}(m)$ (hence, $\hat{y}(m) \in \Pi_a$). Indeed, otherwise

some $y_{-j}(m)$ equals 0: the critical point of f_ε. Then there are subsequences of indices n_u and q_u such that $a^{n_u}_{-q_u} \to 0$, as $u \to \infty$. Therefore, the terms in (5.5) corresponding to the points $y_{-j} = a^{n_u}_{-q_u}$ tend to infinity. Hence, $\beta(\hat{a}, \hat{a}^{n_u}) \to \infty$ (Corollary 5.4), - a contradiction. Statement (5.9) is proved. □

Proof of (5.10). Fix a $m = 1, \dots, l$. Everywhere in the proof, whenever the contrary is not specified, f_ε^{-1} denotes the inverse branch $D_\sigma(a(\varepsilon)) \to D_\sigma(a(\varepsilon))$ fixing $a(\varepsilon)$: its iterates converge to $a(\varepsilon)$ uniformly on $D_\sigma(a(\varepsilon))$, by (5.16). Recall that $a^n_{-\nu_k(\hat{a}^n)} \in D_\sigma(a(\varepsilon))$ for all $k = 1, \dots, l$ and $n \in \mathbb{N}$, by (5.20). Consider the following auxiliary backward orbits. If $m < l$, we set

$$\hat{b}^n = (a^n_{-\nu_m(\hat{a}^n)}, a^n_{-\nu_m(\hat{a}^n)-1}, \dots, a^n_{-\nu_{m+1}(\hat{a}^n)}, f_\varepsilon^{-1}(a^n_{-\nu_{m+1}(\hat{a}^n)}), f_\varepsilon^{-2}(a^n_{-\nu_{m+1}(\hat{a}^n)}), \dots), \quad (5.33)$$

$$\hat{c}^n = \hat{f}_\varepsilon^{\ \nu_m(\hat{a}^n)-\nu_{m+1}(\hat{a}^n)}(\hat{b}^n) = (a^n_{-\nu_{m+1}(\hat{a}^n)}, f_\varepsilon^{-1}(a^n_{-\nu_{m+1}(\hat{a}^n)}), \dots). \text{ If } m = l, \text{ we set}$$

$$\hat{b}^n = \hat{f}_\varepsilon^{\ -\nu_l(\hat{a}^n)}(\hat{a}^n) = (a^n_{-\nu_l(\hat{a}^n)}, a^n_{-\nu_l(\hat{a}^n)-1}, \dots), \ \hat{c}^n = \hat{a}.$$

By construction, $\hat{b}^n, \hat{c}^n \in L(\hat{a})$. We use the formula

$$\Sigma_m = \beta_{\hat{b}^n} - \beta_{\hat{c}^n}. \quad (5.34)$$

This follows from definition, see (5.10), and formula (5.4) applied to the following pairs of orbits: $\hat{x} = \hat{a}$, $\hat{y} = \hat{b}^n$; $\hat{x} = \hat{a}$, $\hat{y} = \hat{c}^n$. We show that

$$\beta_{\hat{b}^n} \to \beta_{\hat{y}(m)}, \ \beta_{\hat{c}^n} \to \beta_{\hat{a}}, \text{ as } n \to \infty. \quad (5.35)$$

This together with (5.34) and the equality $\beta(\hat{a}, \hat{y}(m)) = \beta_{\hat{y}(m)} - \beta_{\hat{a}}$ implies (5.10).

For every $n \in \mathbb{N}$ one has $\hat{c}^n \in L(\hat{a}, D_\sigma(a(\varepsilon)))$, and $c^n_0 = a^n_{-\nu_{m+1}(\hat{a}^n)} \to a(\varepsilon)$, as $n \to \infty$ ($\hat{c}^n = \hat{a}$, if $m = l$), by construction and (5.31). Hence, $\hat{c}^n \to \hat{a}$, $\beta_{\hat{c}^n} \to \beta_{\hat{a}}$, as $n \to \infty$. To prove that $\beta_{\hat{b}^n} \to \beta_{\hat{y}(m)}$, we fix a neighborhood $V = V(a(\varepsilon)) \subset \overline{\mathbb{C}}$ such that the local leaf $L(\hat{y}(m), V)$ is univalent over V. We show that

$$\hat{b}^n \in L(\hat{y}(m), V) \text{ for every } n \text{ large enough.} \quad (5.36)$$

Proof of (5.36). If $m < l$, we set

$$\mu(\hat{a}^n) = k_{r(m+1)-1}(\hat{a}^n), \text{ see (5.21). If } m = l, \text{ we set } \mu(\hat{a}^n) = k_s(\hat{a}^n).$$

One has $\nu_m(\hat{a}^n) = j_{r(m)}(\hat{a}^n) < \mu(\hat{a}^n) \le \nu_{m+1}(\hat{a}^n) = j_{r(m+1)}(\hat{a}^n)$, by (5.21),

$$a^n_{-\mu(\hat{a}^n)}, \dots, a^n_{-\nu_{m+1}(\hat{a}^n)} \in D_\sigma(a(\varepsilon)), \quad (5.37)$$

by (5.22). Hence, by (5.33),

$$b^n_{-j} \in D_\sigma(a(\varepsilon)), \ b^n_{-j-1} = f_\varepsilon^{-1}(b^n_{-j}) \text{ for every } j \ge q(n) = q_m(n) = \mu(\hat{a}^n) - \nu_m(\hat{a}^n). \quad (5.38)$$

Claim 1. *The above numbers $q(n)$ are uniformly bounded from above.*
Proof There exists a $\sigma' > 0$ such that

$$b^n_{-j} = a^n_{-j-\nu_m(\hat{a}^n)} \notin D_{\sigma'}(a(\varepsilon)) \text{ for every } n \in \mathbb{N} \text{ and } 0 < j < q(n). \quad (5.39)$$

Indeed, the converse would mean that there exist sequences of indices $n_u, p_u \in \mathbb{N}$, $\nu_m(\hat{a}^{n_u}) < p_u < \mu(\hat{a}^{n_u})$, such that $a^{n_u}_{-p_u} \to a(\varepsilon)$, as $u \to \infty$. Note that the latter lower and upper bounds for p_u are the moments of leaving $D_\sigma(a(\varepsilon))$ and returning to $D_\sigma(a(\varepsilon))$ of the backward orbit $\hat{a}^{n_u}$. Without loss of generality we consider that the holomorphic branch $f_\varepsilon^{-1} : D_\sigma(a(\varepsilon)) \to D_\sigma(a(\varepsilon))$ contracts the distances. Then for every $\hat{y} \in \Pi_a$ with $y_{-1} = -a(\varepsilon)$ the local minima for $dist(y_{-j}, a(\varepsilon))$ are achieved at $j \in J(\hat{y})$. Hence, after passing to a subsequence, the above p_u may be chosen equal to $j_r(\hat{a}^{n_u})$, $r(m) < r < r(m+1)$ ($r > r(m)$, if $m = l$): $a^{n_u}_{-j_r(\hat{a}^{n_u})} \to a(\varepsilon)$, as $n \to \infty$, - a contradiction to the definition of the numbers $r(m)$, see (5.31). This proves (5.39). It implies that the number $q(n) - 1$ is no greater than the number $d(\hat{a}^n)$ from (5.27) defined with σ replaced by σ'. This together with (5.28) proves the claim. □

Without loss of generality we assume that $q = q(n)$ is independent on n, passing to a subsequence. Let V be the neighborhood of $a(\varepsilon)$ from (5.36). For the proof of (5.36) we have to show that for every n large enough and each $j \in \mathbb{N}$ the inverse branch $f_\varepsilon^{-j} : y_0(m) \mapsto y_{-j}(m)$ (which is single-valued on V by construction) sends b^n_0 to b^n_{-j}. Indeed, for every $j \in \mathbb{N} \cup 0$ one has $b^n_{-j} \to y_{-j}(m)$, as $n \to \infty$, by (5.32) and (5.33). This implies the latter inverse branch statement for each n large enough and every $j \leq q$. For each $j > q$ the attractive inverse branch $f_\varepsilon^{-1} : D_\sigma(a(\varepsilon)) \to D_\sigma(a(\varepsilon))$ sends b^n_{-j} to b^n_{-j-1} for every $n \in \mathbb{N}$, by (5.33), (5.37) and (5.19). The same holds true for b^n_{-j} replaced by $y_{-j}(m)$, passing to the limit. This proves (5.36). □

One has $\hat{b}^n \to \hat{y}(m)$, $\beta_{\hat{b}^n} \to \beta_{\hat{y}(m)}$, as $n \to \infty$, by (5.36) and since $b^n_0 = a^n_{-\nu_m(\hat{a}^n)} \to y_0(m) = a(\varepsilon)$. This proves (5.35) and (5.10). The proof of Lemma 5.5 is complete. □

5.4 Accumulation set of a horosphere. Proof of (5.3)

For the proof of (5.3) we have to prove the two following statements:

$$\text{the accumulation set of the horosphere } S = S_{\hat{a},0} \text{ is contained in } SB_\varepsilon; \tag{5.40}$$

$$\text{the horosphere } S \text{ accumulates at least to the whole set } SB_\varepsilon. \tag{5.41}$$

Recall that we measure heights of horospheres with respect to the Euclidean metric of $\mathbb{C}$ lifted to the leaves of $\mathcal{A}_{f_\varepsilon}$.

Proof of (5.41). For every $\hat{y} \in \Pi_a$ and $n \in \mathbb{N}$ set $\alpha^n(\hat{y}) = (\hat{f}^n(\hat{y}), \beta(\hat{a}, \hat{y})) \in \mathcal{H}_{f_\varepsilon}$. One has $\alpha^n(\hat{y}) \in S$, since $(\hat{f}^n(\hat{y}), \beta(\hat{a}, \hat{f}^n(\hat{y})) \in S$, see (2.5), and $\beta(\hat{a}, \hat{y}) = \beta(\hat{a}, \hat{f}^n(\hat{y}))$ (the invariance of basic cocycle); $\alpha^n(\hat{y}) \to \alpha(\hat{y}) = (\hat{a}, \beta(\hat{a}, \hat{y}))$, as $n \to +\infty$ ($\hat{f}^n(\hat{y}) \to \hat{a}$, by Corollary 1.36). Hence, S accumulates at least to the horospheres through all the points $\alpha(\hat{y})$, $\hat{y} \in \Pi_a$. The set SB_ε is the closure of the union of the latter horospheres. This proves (5.41). □

As it is shown below, statement (5.40) is implied by the following proposition.

Proposition 5.7 *Let $\hat{a}^n$, l, ν_m, $\hat{y}(m)$ be the same, as in Lemma 5.5. Then $\hat{a}^n \to \hat{y}(1)$ in $\mathcal{A}_{f_\varepsilon}$, as $n \to \infty$.*

Corollary 5.8 *For every sequence of distinct points $(\hat{a}^n, h_n) \in S$, $\hat{a}^n \in \Pi_a$, converging in $\mathcal{H}_{f_\varepsilon}$ to a point $(\hat{y}, h) \in \mathcal{H}_{f_\varepsilon}$ with $(\pi_0|_{L(\hat{y})})'(\hat{y}) \neq 0$, $h \in \mathbb{R}$, the latter limit is contained in SB_ε.*

In the proof of Proposition 5.7 and Corollary 5.8 we use the following

Claim 2. *In the conditions of Lemma 5.5 for every n large enough and each $j \leq \nu_2(\hat{a}^n)$ one has $a^n_{-j} = y_{-j}(1)$. For every $m = 1, \dots, l$ one has $a^n_{-\nu_m(\hat{a}^n)-j} \to y_{-j}(m)$, as $n \to \infty$, uniformly in $0 \leq j \leq \nu_{m+1}(\hat{a}^n) - \nu_m(\hat{a}^n)$.*

Proof Let $q(n) = q_m(n)$ be the numbers from (5.38). They are uniformly bounded in m and n, by Claim 1 at the same place. Let $q = \max_{m,n} q_m(n)$. For every $q \leq j \leq \nu_{m+1}(\hat{a}^n) - \nu_m(\hat{a}^n)$ one has $a^n_{-\nu_m(\hat{a}^n)-j} \in D_\sigma(a(\varepsilon))$, by (5.37). Hence, for each $q < j \leq \nu_{m+1}(\hat{a}^n) - \nu_m(\hat{a}^n)$ the point $a^n_{-\nu_m(\hat{a}^n)-j}$ is obtained from $a^n_{-\nu_m(\hat{a}^n)-j+1}$ by applying the attractive inverse branch $f_\varepsilon^{-1} : D_\sigma(a(\varepsilon)) \to D_\sigma(a(\varepsilon))$, by (5.19). The same branch transforms the limit $y_{-j+1}(m)$ of the latter to the limit $y_{-j}(m)$ of the former, see (5.30). Therefore, the convergence $a^n_{-\nu_m(\hat{a}^n)-j} \to y_{-j}(m)$, which is uniform in $j \leq q$, is also uniform in all $j \leq \nu_{m+1}(\hat{a}^n) - \nu_m(\hat{a}^n)$. This proves the uniform convergence statement of the claim. The first statement of the claim holds true by the uniform convergence and since $a^n_0 = y_0(m) = a(\varepsilon)$. □

Proof of Corollary 5.8. Let $\alpha^n = (\hat{a}^n, h_n)$ be the same, as in the corollary. Then $h_n = \beta(\hat{a}, \hat{a}^n)$, since $\alpha^n \in S$ by assumption and by (2.5). Set

$$u_n = \max\{j \mid a^n_{-j} = a(\varepsilon)\}, \ \hat{b}^n = \hat{f}^{-u_n}(\hat{a}^n):$$

$$b^n_{-1} = -a(\varepsilon), \ \beta(\hat{a}, \hat{b}^n) = \beta(\hat{a}, \hat{a}^n) = h_n \to h, \text{ as } n \to \infty, \tag{5.42}$$

by the invariance of basic cocycle. Thus, the sequence $\hat{b}^n$ satisfies the conditions of Lemma 5.5. Let $l \in \mathbb{N}$ and $\hat{y}(m) \in \Pi_a$ be the same as in the lemma. Then $\hat{b}^n \to \hat{y}(1)$ in $\mathcal{A}_{f_\varepsilon}$ (Proposition 5.7),

$$(\hat{b}^n, h_n) \to (\hat{y}(1), h) \text{ in } \mathcal{H}_{f_\varepsilon}, \ h = \sum_{m=1}^{l} \beta(\hat{a}, \hat{y}(m)) \tag{5.43}$$

(Corollary 5.6). One has $(\hat{b}^n, h_n) = (\hat{b}^n, \beta(\hat{a}, \hat{b}^n)) \in S$, by (5.42) and (2.5). Let us consider the three following possible cases:

Case 1): $u_n = 0$ for all n. Then $\hat{a}^n = \hat{b}^n$. One has $l \geq 2$. Indeed, otherwise, if $l = 1$, then $\hat{b}^n = \hat{y}(1)$ for every n large enough (Claim 2: here $\nu_2 = \nu_{l+1} = +\infty$). But $(\hat{b}^n, h_n) \in S$ are distinct points, by assumption. Hence, $\hat{b}^n$ are also distinct, since the restriction to each horosphere of the π_{hyp}- projection is injective. This contradicts the equality $\hat{b}^n = \hat{y}(1)$. Thus, $l \geq 2$, $\hat{y} = \hat{y}(1)$,

$$S_{\hat{y},h} = S_{\hat{a},h'}, \ h' = h - \beta(\hat{a}, \hat{y}) = \sum_{m=2}^{l} \beta(\hat{a}, \hat{y}(m)), \tag{5.44}$$

by (5.43) and (2.5). The latter horosphere is contained in SB_ε by (5.2). This proves the statement of the corollary.

Case 2): u_n is bounded. Without loss of generality we consider that $u_n = const = u$, passing to a subsequence. Then $l \geq 2$, as in the above case, and $\hat{y} = \hat{f}^u(\hat{y}(1))$. Statement (5.44) holds true again, since

$$h - \beta(\hat{a}, \hat{y}) = \sum_{m=1}^{l} \beta(\hat{a}, \hat{f}^u(\hat{y}(m))) - \beta(\hat{a}, \hat{f}^u(\hat{y}(1)) = h',$$

by (5.43) and the equalities $\beta(\hat{a}, \hat{f}^u(\hat{y}(m))) = \beta(\hat{a}, \hat{y}(m))$ (the invariance of basic cocycle). This proves the corollary.

Case 3): $u_n \to \infty$ (after passing to a subsequence). Then

$$\hat{a}^n = \hat{f}^{u_n}(\hat{b}^n) \to \hat{a} \text{ in } \mathcal{A}_{f_\varepsilon}, \text{ as } n \to \infty. \tag{5.45}$$

Indeed, let $V \subset \mathbb{C}$ be a neighborhood of $a(\varepsilon)$ over which the local leaf $L(\hat{y}(1), V)$ is univalent, U be a smaller neighborhood: $\overline{U} \subset V$. Then the local leaves $L(\hat{b}^n, U)$ are univalent over U, whenever n is large enough, by the convergence $\hat{b}^n \to \hat{y}(1)$ in $\mathcal{A}_{f_\varepsilon}$ and Proposition 1.33. This together with Corollary 1.36 implies (5.45). Thus, $(\hat{a}^n, h_n) \to (\hat{a}, h)$, $h_n \in B_\varepsilon$, and hence, $h \in B_\varepsilon$, since B_ε is closed. Therefore, $(\hat{a}, h) \in SB_\varepsilon$. The proof of the corollary is complete. □

Proof of (5.40) modulo Proposition 5.7. Let the horosphere S accumulate to some horosphere $S' \subset \mathcal{H}_{f_\varepsilon}$, $L' \subset \mathcal{A}_{f_\varepsilon}$ be the corresponding affine leaf. The leaf L' is contained in $\mathcal{A}'_{f_\varepsilon}$, and it is dense there (Proposition 1.38). Hence, there exists a point $\hat{y} \in L'$ such that $\pi_0(\hat{y}) = a(\varepsilon)$ and $(\pi_0|_{L'})'(\hat{y}) \neq 0$, since $\Pi_a \neq \emptyset$. Let h be the height of the horosphere S' over $\hat{y}$. Then $(\hat{y}, h)$ is a limit of a sequence of distinct points $(\hat{a}^n, h_n) \in S$, $\hat{a}^n \in \Pi_a$. Hence, $(\hat{y}, h) \in SB_\varepsilon$ and $S' \subset SB_\varepsilon$, by Corollary 5.8. This proves (5.40). □

Proof of Proposition 5.7. One has $\hat{a}^n \to \hat{y}(1)$ in $\mathcal{N}_{f_\varepsilon}$, by (5.30). Let us prove the convergence in $\mathcal{A}_{f_\varepsilon}$. Fix a $N \in \mathbb{N}$ and a neighborhood V of $y_{-N}(1)$ such that the local leaf $L(\hat{f_\varepsilon}^{-N}(\hat{y}(1)), V)$ is univalent over V, and its arbitrary smaller neighborhood V', $\overline{V'} \subset V$. Let us show that the local leaf $L(\hat{f_\varepsilon}^{-N}(\hat{a}^n), V')$ is also univalent over V', whenever n is large enough. This together with Proposition 1.33 proves Proposition 5.7.

Fix a neighborhood $W = W(a(\varepsilon)) \subset \mathbb{C}$ such that for every $m = 2, \dots, l$ the local leaf $L(\hat{y}(m), W)$ is univalent over W. Fix an arbitrary smaller neighborhood $U = U(a(\varepsilon))$, $\overline{U} \subset W$. For every $k \in \mathbb{N} \cup 0$ and each $\hat{x} \in \mathcal{A}^l_{f_\varepsilon}$ set

$$f^{-k}_{\varepsilon,\hat{x}} = \text{ the germ of the inverse branch } f_\varepsilon^{-k} \text{ sending } x_0 \text{ to } x_{-k}. \text{ Set}$$

$$g_1^{-k} = f^{-k}_{\varepsilon,\hat{f}^{-N}(\hat{y}(1))},\ g_m^{-k} = f^{-k}_{\varepsilon,\hat{y}(m)} \text{ for every } m \geq 2,$$

$$g^{-k}_{N,n} = f^{-k}_{\varepsilon,\hat{f}^{-N}(\hat{a}^n)},\ \phi^{-k}_{m,n} = f^{-k}_{\varepsilon,\hat{f}^{-\nu_m(\hat{a}^n)}(\hat{a}^n)} \text{ for every } n \in \mathbb{N},\ m = 1, \dots, l. \text{ One has:}$$

$$g_1^{-k} \text{ is holomorphic on } V \text{ and } g_1^{-k} \to a(\varepsilon) \text{ uniformly on } \overline{V'}, \text{ as } k \to \infty, \tag{5.46}$$

$$g_m^{-k} \text{ is holomorphic on } W \text{ and } g_m^{-k} \to a(\varepsilon) \text{ uniformly on } \overline{U}, \text{ for every } m \geq 2, \tag{5.47}$$

by the univalence of the local leaves $L(\hat{f_\varepsilon}^{-N}(\hat{y}(1)), V)$, $L(\hat{y}(m), W)$ and the Shrinking Lemma.

Claim 3. *For every n large enough and $k \in \mathbb{N}$ the function $g^{-k}_{N,n}$ is holomorphic on V'.*
Proof For every n large enough one has

$$a^n_{-N-k} = y_{-N-k}(1), \text{ thus, } g^{-k}_{N,n} = g_1^{-k}, \text{ for every } k \leq \nu_2(\hat{a}^n) - N, \tag{5.48}$$

$$g_1^{N-\nu_2(\hat{a}^n)}(\overline{V'}) \subset U,\ g_m^{\nu_m(\hat{a}^n)-\nu_{m+1}(\hat{a}^n)}(\overline{U}) \subset U \text{ for every } m \geq 2, \tag{5.49}$$

$$g_m^{-k} = \phi^{-k}_{m,n} \text{ for every } m \geq 2 \text{ and } k \leq \nu_{m+1}(\hat{a}^n) - \nu_m(\hat{a}^n). \tag{5.50}$$

Statement (5.48) follows from the first statement of Claim 2. Statement (5.49) follows from (5.46), (5.47) and (5.7). Statement (5.50) follows from the second statement of Claim 2. Fix

an arbitrary n satisfying (5.48)-(5.50). Let us prove the holomorphicity of $g_{N,n}^{-k}$ on V'. We prove this for every $k > \nu_l(\hat{a}^n)$: this implies the holomorphicity of $g_{N,n}^{-k'} = f^{k-k'} \circ g_{N,n}^{-k}$ for smaller k'. One has $g_{N,n}^{N-\nu_2(\hat{a}^n)} = g_1^{N-\nu_2(\hat{a}^n)}$, by (5.48). Hence,

$$g_{N,n}^{-k} = \phi_{l,n}^{\nu_l(\hat{a}^n)-k} \circ \phi_{l-1,n}^{\nu_{l-1}(\hat{a}^n)-\nu_l(\hat{a}^n)} \circ \dots \circ \phi_{2,n}^{\nu_2(\hat{a}^n)-\nu_3(\hat{a}^n)} \circ g_{N,n}^{N-\nu_2(\hat{a}^n)}$$

$$= g_l^{\nu_l(\hat{a}^n)-k} \circ g_{l-1}^{\nu_{l-1}(\hat{a}^n)-\nu_l(\hat{a}^n)} \circ \dots \circ g_2^{\nu_2(\hat{a}^n)-\nu_3(\hat{a}^n)} \circ g_1^{N-\nu_2(\hat{a}^n)},$$

and the latter composition is well-defined and holomorphic on V', by (5.49) and (5.50). This together with the previous discussion proves Claim 3. $\square$

Claim 3 implies the univalence of the local leaf $L(\hat{f}_\varepsilon^{-N}(\hat{a}^n), V')$ over V' for every n large enough. Together with Proposition 1.33, this proves Proposition 5.7. $\square$

5.5 Perturbations. Proof of the Addendum to Theorem 1.57

The proof of the Addendum to Theorem 1.57 is analogous to that of the theorem itself with minor modifications listed below. Most of them concern the proof of inequality (5.28). We fix a $\varepsilon_0 < \frac{1}{4}$, $\varepsilon_0 \neq 0$ (that may be equal to -2). Let σ and δ be the numbers from (5.16)-(5.18), both corresponding to $\varepsilon = \varepsilon_0$. We show that there exists a $\Delta > 0$ such that (5.28) holds true for every complex value $\varepsilon \neq -2$, $|\varepsilon - \varepsilon_0| < \Delta$, with δ replaced by $\frac{\delta}{2}$. To do this, we use the three following well-known facts:

- the closure of a (super-)attracting basin of a rational function of a given degree depends lower-semicontinuously on its coefficients in the Hausdorff topology;

- the Julia set of a complex polynomial is the boundary of the super-attracting basin of infinity;

- there is a complex neighborhood of the interval $0 < \varepsilon < \frac{1}{4}$ where the Julia set $J(f_\varepsilon)$ is a Jordan curve separating the super-attracting basin of infinity from the attracting basin of a finite attracting fixed point.

For every given $\varepsilon_0 < 0$ ($0 < \varepsilon_0 < \frac{1}{4}$) and each $\sigma' > 0$ there exists a $\Delta > 0$ such that for every $\varepsilon \in \mathbb{C}$, $|\varepsilon - \varepsilon_0| < \Delta$, the corresponding inclusion from Lemma 5.2 holds true with $J(f_\varepsilon)$ replaced by

$$J_{\varepsilon,\sigma'} = J(f_\varepsilon) \setminus (D_{\sigma'}(a(\varepsilon)) \cup D'_{\sigma'}) :$$

$$J_{\varepsilon,\sigma'} \subset D_{a(\varepsilon)} \text{ (respectively, } J_{\varepsilon,\sigma'} \subset \mathbb{C} \setminus \overline{D_{a(\varepsilon)}}), \tag{5.51}$$

where $D'_{\sigma'}$ is the corresponding domain (5.15). This follows from Lemma 5.2 and the above semicontinuity statement applied to the basin of infinity (respectively, the basin of a finite attracting fixed point). Let for simplicity $0 < \varepsilon_0 < \frac{1}{4}$: then for every $\hat{y} \in \Pi_a$ all the terms in the corresponding formula (5.5) are positive. If $\varepsilon \in D_\Delta(\varepsilon_0)$ is not real, then, in general, some terms in (5.5) may be negative. But this may happen only to those corresponding to $y_{-j} \in D'_{\sigma'} \cup D_{\sigma'}(a(\varepsilon))$. This follows from (5.51), as in the proof of Corollary 5.3. The other terms are no less than δ, see (5.18), if $\varepsilon = \varepsilon_0$, and greater than $\frac{2\delta}{3}$, whenever ε is complex and $\varepsilon - \varepsilon_0$ is small enough. Elementary estimates show that if $\varepsilon - \varepsilon_0$ is small enough, then the sum of the latter terms strongly dominates that of the former ones so that inequality (5.28) holds with δ replaced by $\frac{\delta}{2}$. The rest of the proof of Theorem 1.57 applies with obvious changes.

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